MARY S. KABIRI AND NANCY L. SMITH

Turning Traditional Textbook Problems into Open-Ended Problems

THE EQUITY PRINCIPLE CONTAINED IN *Principles and Standards for School Mathematics* states that we should have ". . . high expectations and strong support for all students" (NCTM 2000, p. 12). Often, when teachers plan instruction for their students, they focus on the middle achievers. However, to truly provide equitable instruction, students with special interests or talents in mathematics may need additional resources to challenge and engage them (NCTM 2000, pp. 12–13) while those students who are lacking prerequisite skills need extra support. To address the difficulty of meeting the needs of *all* learners in their class, many teachers have found

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MENU

Side Dishes		**Main Dishes**		**Desserts**	
Taco Salad	$3.50	Chicken and Rice	$6.95	Flan	$3.25
Quesadillas	$2.25	Beef Fajitas	$8.95	Baked Apples	$2.45
Cheese Nachos	$1.50	Chicken Fingers	$4.50	Empanadas	$3.15
Baked Potato	$1.75	Chimichangas	$5.95		
Cole Slaw	$1.00	Beef Burritos	$4.95		

Traditional form: Randy, Becky, CJ, Lauren, and Ty go to eat dinner at their favorite restaurant. Ty orders quesadillas, beef fajitas, and flan. CJ has chicken fingers and does not order a side dish or dessert. How much do these two meals cost? (Houghton Mifflin 2002, p. 54).

Open-ended form: Randy has $13.00 to spend at his favorite restaurant. He wants to order one main dish, two side dishes, and one dessert. He knows he will spend $1.50 on video games while he waits for his order. Find three different meals that Randy could choose. Show your calculations and explain how you thought about the problem.

Fig. 1 A Number and Operations problem shown in both a traditional and open-ended form

success by "basing instruction on problems and activities that invite different solution approaches and many levels of solution so that less talented students can participate in the task with more talented individuals and all can experience individual success . . ." (Bley and Thornton 1994, p. 158). When teachers present these types of problems, they not only support the high achievers but also communicate high expectations and provide opportunities for higher-level thinking for all students in the class. This strategy promotes good classroom management in that it provides enrichment or sponge activities for students who finish early and are ready for more challenge while giving slower finishers the time they need as well. Traditional textbook problems do not always lend themselves to multiple solutions or solution strategies. However, many problems can be made more open-ended and accessible to a wide variety of student ability levels with minimum effort. In this article, we will show how this transformation was accomplished in a mixed-ability seventh-grade classroom.

A mathematical problem involves a situation for which the solution is not immediately obvious. A traditional problem gives a set of constraints or conditions, and strategies are applied to obtain a desired result. See this problem, for example:

> Insert parentheses to make the sentence true: $5 + 2 \times 3 - 7/2 = 7$.

An open-ended problem has an additional dimension—more than one answer is possible. For example:

> Use four 4s and any of the four fundamental operations or parentheses to write mathematical sentences that make the number 8.

Ocasionally, this definition of *open-ended* is broadened to include problems for which different approaches or strategies lead to the correct single result. It is the *approach* that is open-ended. For example, "Find several different ways to calculate 38 + 15 + 42 + 18 mentally" asks for an open-ended approach. However, if the teacher defines one approach as "best," then the open-endedness is lost (Shimada 1997, p. 1).

One way that teachers can easily include more open-ended problem solving is to take traditional problems from their textbooks and adapt one or more in a lesson so that all students in the class have access to enrichment beyond the regular textbook lesson. Current textbook publishers seem to be doing a much better job of providing open-ended problems in their lessons. However, traditional problems are also represented and may be adapted. To demonstrate this approach, we took five traditional problems from a 2002 textbook, one each from the first five NCTM Content Standards. We adapted them so that they would be more open-ended and asked some seventh graders to solve them. These seventh graders were in a heterogeneous mathematics class from Lewis and Clark Middle School in Jefferson City, Missouri. What follows are the problems, their adaptations, and some student solutions.

Number and Operations

THE TRADITIONAL FORM OF THE MENU PROBLEM shown in **figure 1** may be found in most textbooks. However, it becomes more open-ended when students may select the items to purchase and are directed to find more than one solution. Providing a

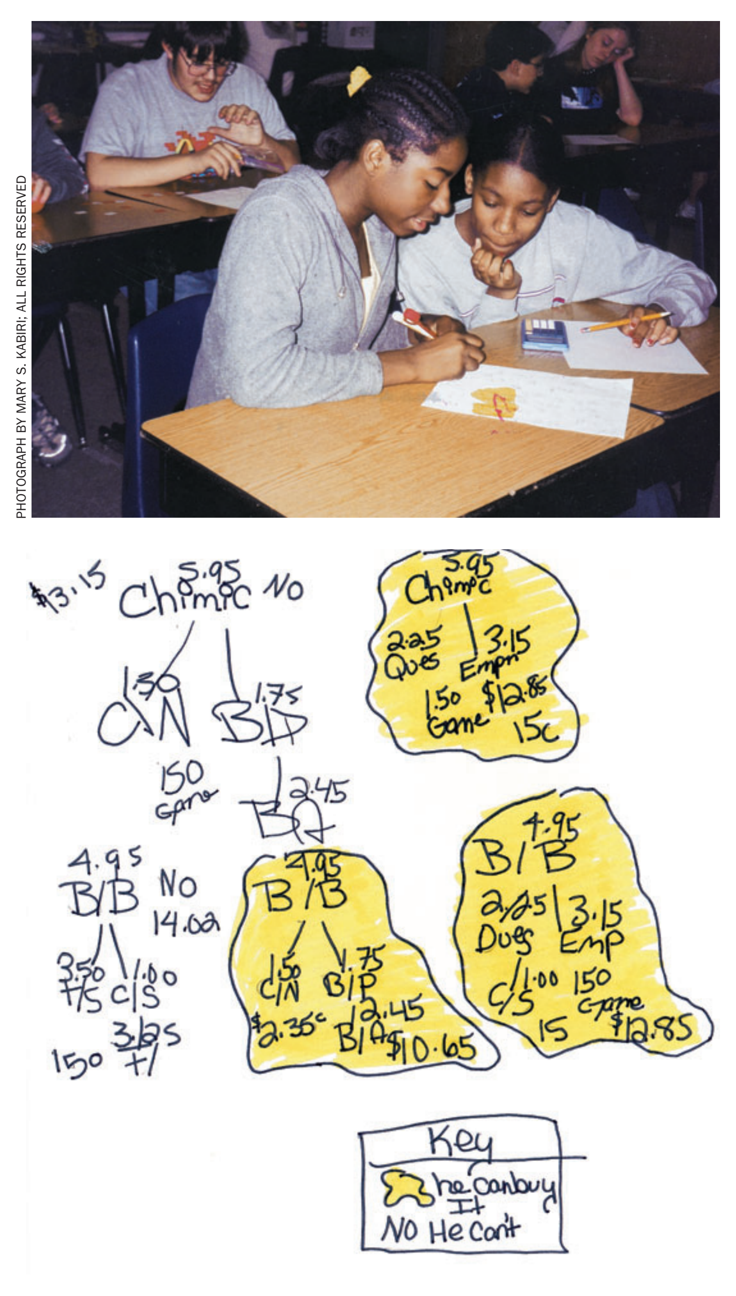

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Fig. 2 Jasma and Keona solve the menu problem by using tree diagrams.

budget and a menu with choices also gives students a more real-life problem to solve. Not only are students using problem-solving skills, they are also getting valuable practice adding multiple combinations of decimals. They are often eager to select a meal that they would like to order or enjoy the challenge of getting the most for their money.

Traditional form: In a game, a player's scores on five successive turns were +8, –11, +7, –7, +6. After which two turns was the player's total score the same? How many points were scored altogether during those five turns? (Houghton Mifflin 2002, p. 220).

Open-ended form: You have a set of integer cards from –9 to 9 in a bag, a six-sided die, and a set of plus and minus (+/–) operation cards. Shake the bag, and draw out 8 integer cards. Roll the die. This is your target number. Use the 8 drawn integer cards and any of the operation cards to make integer sentences that use addition and subtraction and which total the number on your die. Record your sentence, return the cards to the pile of 8, and make as many sentences as you can.

Fig. 3 An Algebra problem shown in both a traditional and open-ended form

At first, several students commented that they thought $13.00 was a lot of money. They thought that they could randomly select any combination and it would fit within their budget. However, once they got started, they found that several combinations cost more than $13.00. When Meghan and Lauren saw that a random selection of items was not working, they decided to select the cheapest item from each category and were then successful.

When given the open-ended opportunity, some students will go beyond the teacher's expectations. Jasma and Keona decided to use tree diagrams to solve this problem (see **fig. 2**), having studied this strategy several months earlier. They filled three pages with tree diagrams to work out all the possible combinations, calculated the totals, and highlighted all the combinations that Randy could buy for $13.00. The Problem Solving Standard recommends that through "problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence . . ." (NCTM 2000, p. 52). Open-ended problems like this one involving a menu encourage students to develop those characteristics.

Algebra

THE PROBLEM IN **FIGURE 3** IS TRADITIONAL BEcause there is only one solution and primarily one way to solve it. Using cards or dice to have students generate problems is one way to develop an infinite number of integer sentences. Challenging students to find as many solutions as they can also encourages them to use their creativity and be persistent.

Using integer cards and a die put the problem in a game format and also encouraged the use of the guess-and-check strategy, since the cards could be arranged and rearranged to form the number sentences. When in an open-ended setting, some students meet the minimum requirements, whereas others will go beyond our expectations for the problem, therefore receiving

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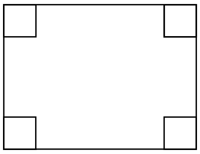
Fig. 4 Rachael and David solve the integer problem.

Traditional form: Give the correct name for each polygon.

10. 11. 9-sided figure 12. 12-sided figure



(Houghton Mifflin 2002, p. 420)

Open-ended form: You have been given two congruent right triangles. Your job is to make as many different shapes as you can by joining congruent sides of the two triangles. Draw a picture of each shape, identify it by name, and describe it by writing as many properties as you can (Claus 1989, pp. 57–58).

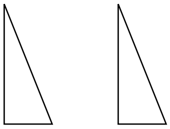


Fig. 5 A Geometry problem shown in both a traditional and open-ended form

much more practice. For example, some students worked primarily with two or three integers at a time forming such sentences as (–5) + (+8) = 3 or (+4) + (+5) + (–1) = 8. However, several students formed lengthy number sentences using all eight integer cards. For example, Aaron and Michael used (+9) + (–8) – (–5) + (–9) + (+5) + (+4) + (+1) – (–1) = 8. They said, "We guessed and checked," and they wrote that "subtracting a negative is the same as adding a positive."

The NCTM's Problem Solving Standard also recommends that students "monitor and reflect on the process of mathematical problem solving" (NCTM 2000, p. 52). Because the problem was also open-ended, several approaches were used and explained by students. Rachael and David also used what they knew about integers to solve the problem but thought about it differently than Aaron and Michael. They said, "We used opposite numbers to get a total of 0 and then added 3 [their target number]" (see **fig. 4**).

Geometry

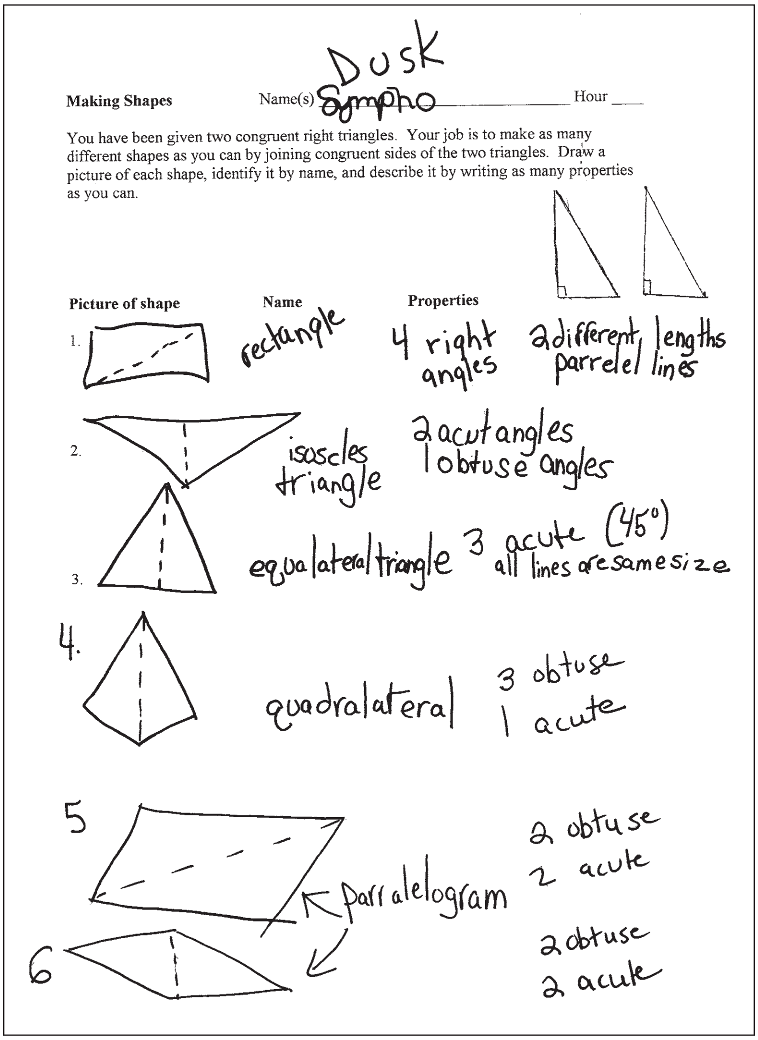
THE TRADITIONAL GEOMETRY PROBLEMS SHOWN in **figure 5** are also very common in texts. However, the seventh-grade students enjoyed the open-ended version that still required them to recognize and name geometric figures but also asked them to go beyond that concept. Manipulating two congruent triangles cut out of an index card supported the guess-and-check problem-solving strategy. The results from this problem clearly illustrate how students can work on a problem at different levels. It was also a great way for the teacher to assess students' fluency with naming and creating two-dimensional figures. Students submitted a wide range of responses, and each pair was able to make three to six figures.

Terry and Ryan formed a parallelogram, triangle, and kite. When describing the figures' properties, they focused on the length of the sides. For example, they made a kite and said that it has two pairs of sides that are the same length but different from the others. Dusk and his partner went a step further. In addition to finding various polygons like the first pair of students did, they also found and named more specialized figures, such as identifying a triangle as being isosceles rather than just a triangle. When looking at the properties of the figures, these students described the types of angles in the figures as being right, acute, or obtuse (see **fig. 6**). Brandon and Justin's descriptions focused on the degree of the angles. For example, they formed a parallelogram, measured it with a protractor, and wrote that it had two 45 degree angles and two 135 degree angles. Each group was able to focus on different properties in their solutions. By having the students complete this open-ended activity, the teacher learned much more about the students' understanding of two-dimensional figures.

Measurement

THE TRADITIONAL FORM OF THE PROBLEM IN **figure 7** requires only applying the volume formula and no critical thinking. To make it more open-ended, we asked students to look for more than one possibility and to make a judgment about which option will be most appropriate.

Principles and Standards recommends that students "apply and adapt a variety of appropriate strategies to solve problems" (NCTM 2000, p. 53). This



Dusk

Making Shapes Name(s) Sympho Hour ____

You have been given two congruent right triangles. Your job is to make as many different shapes as you can by joining congruent sides of the two triangles. Draw a picture of each shape, identify it by name, and describe it by writing as many properties as you can.

Picture of shape	Name	Properties
1.	rectangle	4 right angles 2 different lengths parrelel lines
2.	isoscles triangle	2 acut angles 1 obtuse angles
3.	equalateral triangle	3 acute (45°) all lines are same size
4.	quadralateral	3 obtuse 1 acute
5	parralelogram	2 obtuse 2 acute
6		2 obtuse 2 acute

Fig. 6 Dusk and his partner's solutions to the triangle problem.

problem elicited a variety of strategies and solutions. Although it would have been possible for students to design a dimension using a decimal or a fraction, no one did. They all used whole-number combinations.

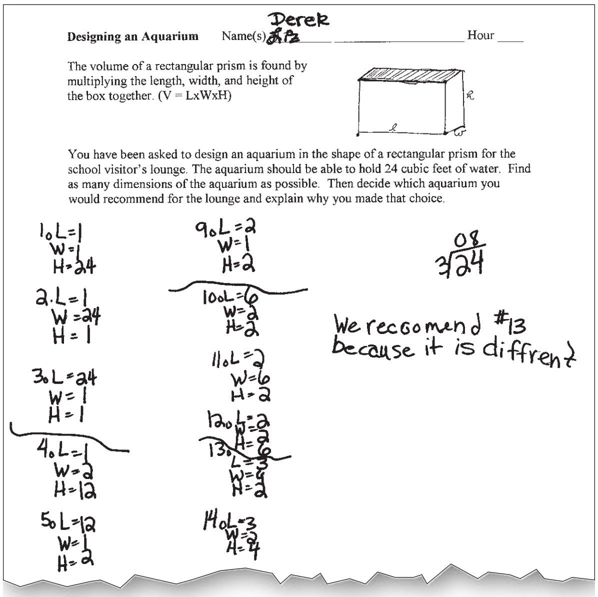
Aaron and Michael drew pictures of the options, then asked the teacher for a ruler. Then they measured the dimensions of one classroom wall to get a life-sized, visual picture before they made their selection. They had chosen an aquarium 8 feet long, 1 foot wide, and 3 feet tall and wanted to be sure "an 8 foot length would make sense." They said they selected the 8 × 1 × 3 "so all the teachers in the lounge won't crowd around and they can walk around it."

Emily and Craig listed possible whole-number dimensions in a table. They said, "We recommend the 4 × 2 × 3 because it is like a box and the water will be deeper than the other aquariums." Derek and Liz demonstrated persistence by making an organized list of thirty-six possibilities. For example, using 3, 4, and 2 could produce an aquarium that is 3 feet long, 4 feet wide, and 2 feet tall; 4 feet long, 2 feet wide, and 3 feet tall; or 2 feet long, 3 feet wide, and 4 feet tall (see **fig. 8**).

Traditional form: A rectangular aquarium is 12 in. wide by 14 in. long by 12 in. high. What is the volume of water needed to fill the aquarium? (Houghton Mifflin 2002, p. 475).

Open-ended form: You have been asked to design an aquarium in the shape of a rectangular prism for the school visitor's lounge. Because of the type of fish being purchased, the pet store recommends that the aquarium should hold 24 cubic feet of water. Find as many different dimensions for the aquarium as possible. Then decide which aquarium you would recommend for the lounge and explain why you made that choice.

Fig. 7 A Measurement problem shown in both a traditional and open-ended form



Derek

Designing an Aquarium Name(s) Liz Hour ____

The volume of a rectangular prism is found by multiplying the length, width, and height of the box together. (V = LxWxH)

You have been asked to design an aquarium in the shape of a rectangular prism for the school visitor's lounge. The aquarium should be able to hold 24 cubic feet of water. Find as many dimensions of the aquarium as possible. Then decide which aquarium you would recommend for the lounge and explain why you made that choice.

1. L=1 W=1 H=24

2. L=1 W=24 H=1

3. L=24 W=1 H=1

4. L=1 W=2 H=12

5. L=12 W=1 H=2

9. L=2 W=1 H=2

10. L=6 W=2 H=2

11. L=2 W=6 H=2

12. L=2 W=2 H=6

13. L=3 W=4 H=2

14. L=3 W=2 H=4

$$3\overline{)24}$$ 08

We recomend #13 because it is diffrent

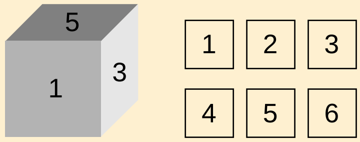
Fig. 8 Derek and Liz's list for the aquarium problem

Brandon and Justin also listed the six whole-number combinations, but they observed that although they found six, they really found three times as many because the numbers could be reordered. Although they did not find all thirty-six possibilities, they were aware that the six number combinations could be rearranged. They also recommended the 4 × 2 × 3 size because "It would give the fish and creatures inside more freedom to move easily."

Data Analysis and Probability

MANY PROBABILITY PROBLEMS OR DISCUSSIONS found in textbooks start like **figure 9**'s illustration of a simple die or sum of two dice. Students are

Traditional form:



The outcomes for tossing this number cube are equally likely. You have the same chance of tossing 1, 2, 3, 4, 5, or 6. This table shows the probabilities of the events of tossing each number when tossing the number cube shown above.

Event	Probability	Event	Probability	Event	Probability
Toss a 1.	P(1) = 1/6	Toss a 2.	P(2) = 1/6	Toss a 3.	P(3) = 1/6
Toss a 4.	P(4) = 1/6	Toss a 5.	P(5) = 1/6	Toss a 6.	P(6) = 1/6

(Houghton Mifflin 2002, p. 516)

Open-ended form: Suppose that you are rolling one green die and one red die and you compute the *sum* on each roll. This chart shows all possible sums.

Green Die

Red Die

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Your task is to design two games for player A and player B. The first game should be a fair game, and the second game should be an unfair game in favor of player A. State the rules of your games, and explain why you think each game is fair or unfair.

Fig. 9 A Data Analysis and Probability activity shown in both a traditional and open-ended form

asked to determine the probability of various outcomes. However, once that procedure is understood, very little creative thought is needed. Simple probability problems of this type may become more open-ended by asking students to look for ways to make a game or experiment fair or unfair. This revision allows students to examine more than one probability concept.

The students found the open-ended form in **figure 9** to be one of the most challenging problems posed. In fact, one group, Rachael and David, was not able to solve it as written but was able to create a fair and unfair game using only one die. They indicated that the game would be fair if player A earned a point for a toss of 1, 2, or 3, and if player B earned a point for a toss of 4, 5, or 6. They also knew that each player had a 3/6 chance of rolling his or her number. To make the game unfair, they assigned player A the numbers 1–5 and player B the number 6.

Brandon and Justin worked for a long time on this problem. They first eliminated all sums of 7, the middle sums that are diagonal in the chart. To be a fair game, player A gets a point if the sum is 6 or below and player B gets a point if the sum is 8 or above. If a 7 is rolled, no one gets a point. They correctly interpreted this situation as being a conditional probability problem by recognizing that after the diagonal of 7s had been eliminated, each player had a 15/30 chance of getting a point. To make the game unfair, they simply allowed player A to get a point for a sum of 7 or above and for player B to get a point for a sum of 6 or below. Player A had a

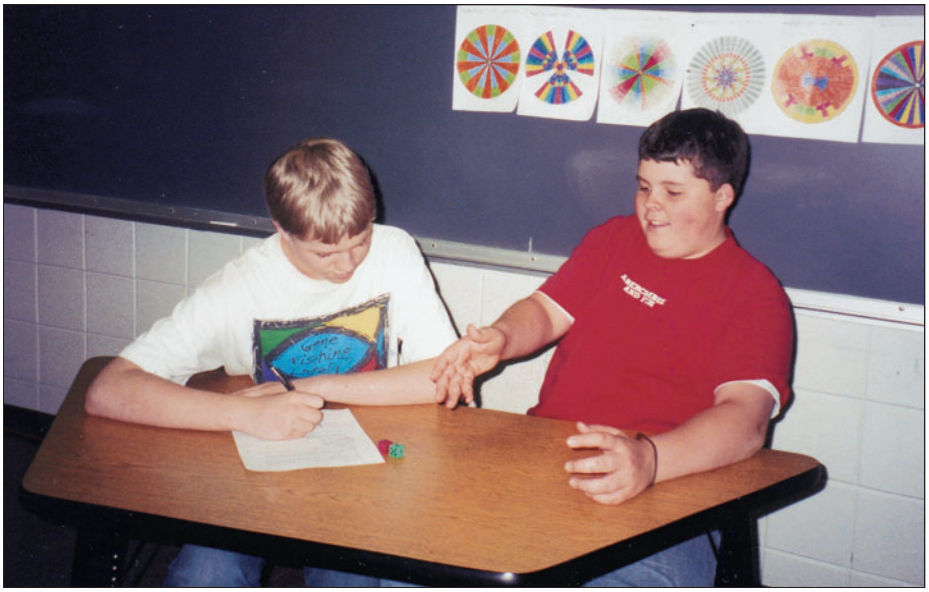


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Fig. 10 **Brandon and Justin try their fair game.**

21/36 chance of getting a point, and player B had a 15/36 chance of getting a point on each roll (see **fig. 10**).

The beauty of open-ended problems like this one is that all students are able to participate with the mathematical concepts at their own level. Each group of students was able to demonstrate an understanding of simple probability and fairness, and those who were able took it to a higher level.

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Conclusions

PROVIDING OPEN-ENDED PROBLEMS helps teachers meet the needs of diverse learners, since all students will benefit. The seventh-grade students in this classroom were of mixed ability. Those who displayed the most persistence or the most elegant solutions were not necessarily the best students. When teachers plan a lesson, asking a few questions will help the process.

- Do I have students who are working below this concept? Who are working above?
- How can I encourage more critical thinking?
- How can I open the tasks to more approaches or solutions to engage more students?

Considering these questions will help us all move toward the NCTM's vision of equity, ". . . high expectations and strong support for all students" (p. 12).

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