And Then There Was Luke:
The Geometric Thinking of a Young Mathematician

Janet M. Sharp and Karen Bush Hoiberg

What might students say about the angles of the pentagonal block shown in **figure 1?** Children might respond in different ways, depending on their abilities and experiences with angles. Some might say that the block "has five angles" after touching each of the corners. Others might observe that "it looks like all the angles are the same size." Perhaps a few would respond as did Luke, a bright fifth grader who is featured in this article. Luke explained that each angle was 108 degrees. You might wonder, How did Luke come to this conclusion? What else does Luke know about angles and shapes, and how could we assess what he knows? This article addresses these questions by telling the story of how Luke's teacher used a specifically designed activity to assess his thinking about angles and shapes. The van Hiele levels of thinking provide a theoretical framework for analyzing Luke's thinking and for offering suggestions for instruction in geometry. The article concludes with a discussion of how Luke and his teacher became partners in studying geometry. This examination is grounded in the student's words, writings, and drawings as he worked to resolve geometric tasks involving angle measurement. Although this article discusses a gifted fifth grader to illustrate the use of the van Hiele theory, the theory can be used to analyze the thinking of, and plan instruction for, any student in any grade.

Janet Sharp, pipurr@iastate.edu, teaches at Iowa State University, Ames, IA 50011. She is interested in developing innovative and effective approaches for teaching mathematics. Karen Hoiberg, khoiberg@dns.ames.k12.ia.us, teaches fifth grade at Fellows Elementary School, Ames, IA 50014. She has received the Presidential Award for Excellence in Science and Mathematics Teaching.

The van Hiele Theory and Angle Measurement

The van Hiele theory (1986) stipulates that with appropriate instruction, learners can pass through five distinct levels of geometric thinking. A student's geometric thinking may be at any level of van Hiele's model for any given concept and is not age dependent. Here, we consider a student's response to a task about angle measurement, during which he demonstrates thinking and reasoning at each of the first three levels. The first three levels of the van Hiele theory are briefly described in **figure 2.**

The activity in **sheet 1** is designed to stimulate level 2 thinking about angle measurement. Students determine the interior angle measurement in degrees for six polygons, without using a protractor but with access to Pentablocks (see **fig. 1**). Pentablocks come in six shapes and can be used to help students develop geometry concepts (Berman, Plummer, and Scheuer 1998).

Luke's teacher, Mrs. Hoiberg, used this specifically designed task in a clinical assessment as part of an assignment for a course that she was taking on teaching geometry. Her task was to select a topic and choose a task to assess the thinking of one of her students on that topic. Mrs. Hoiberg used probes, comments, and questions to draw out Luke's reasoning and to develop a clear picture of his thinking in terms of the van Hiele levels.

Luke, the Mathematician

Luke is an exceptional, well-rounded fifth-grade student and a gifted child mathematician. Hoiberg's characterization of Luke resembles that of a budding Renaissance man:

> Luke is a writer. Luke is a mathematician. Luke is a reader. Luke is an athlete. Luke is a musician. Luke is well liked. Luke is funny. Luke loves learning. Luke is self-motivated. Luke challenges himself in every area every day! Luke is one in a million. I was fortunate to be his fifth-grade teacher.

Before his fifth-grade year, Luke attended a Super Summer mathematics program. There, he studied a variety of mathematical topics with Dr. Leslie Hogben, a mathematician who taught in the program. With her, Luke investigated geometric ideas, such as tessellations, Platonic solids, and constructions in the plane. In so doing, he discovered ideas about angle measurement, such as justifying triangle congruence, comparing angles in tessellating figures, and explaining why the angle sum of a triangle is 180 degrees. During fifth grade, Luke and his classmates applied angle-measurement ideas in map contexts in which they esti-

The Pentablock set

mated, compared, measured, added, and subtracted angle measures. The analysis in the following paragraphs notes the instances when Luke is building on these past mathematical experiences, formulating his own arguments, or engaging in some combination of the two.

Analysis of Luke's Thinking

At first, Luke was challenged when Hoiberg presented the activity in **sheet 1:** "You can't do that without a protractor!" Then Luke recalled an idea that he had discovered while working with Hogben: "I think if you take a polygon and add up the sides and subtract 2, then multiply by 180, you get the amount of all the angles inside that polygon." For the first polygon, which was a regular pentagon, he wrote these equations:

$$5 - 2 = 3 \qquad 180 \times 3 = 540 \qquad 540 \div 5 = 108$$

He then carefully recorded 108 degrees in each of the angles of the pentagon. When asked how he knew that all interior angles were congruent, at first he said, "They all look the same." Hoiberg urged him to reconsider this level 0 answer: "But how can you be certain?" She suspected that Luke could give a more sophisticated explanation than that "they look the same." For children who are at level 0, the level of visual thinking, "looking the same" is sufficient reasoning for congruence.

Luke recognized his teacher's question as an indication that "looking the same" was an insufficient rationale. He physically placed a pentagon block on the drawing and rotated it five times to

Characteristics of the first three levels of thinking in the van Hiele model

Level	General Description	Angle Measurement Examples
0 Visual	"Student identifies, names, compares, and operates on geometric figures according to their appearance."	Decides an angle's size on the basis of "how it looks." Compares angles visually to determine size.
1 Descriptive	"Student analyzes figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically."	Uses protractors to measure; discovers angle relationships, such as that "vertical angles have equal measure."
2 Informal deductive	"Student logically interrelates previously discovered properties/rules by giving or following informal arguments."	Uses properties of parallel lines to explain "why the angle sum of a quadrilateral is 360 degrees" or "why opposite angles of a parallelogram are congruent."

(Adapted from Fuys, Geddes, and Tischler [1988, p. 5])

empirically justify his claim that the shape was equiangular. Luke explained this answer at a higher van Hiele level than before, level 1, after purposeful prodding from his teacher.
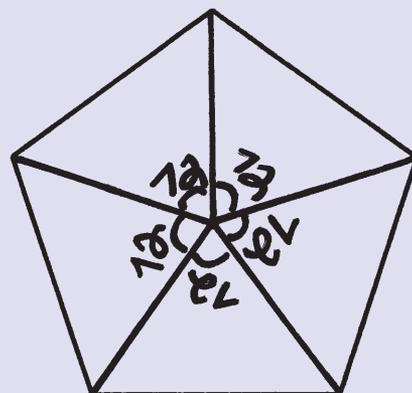
Then Hoiberg steered the conversation back to the rule that Luke had used, $(n - 2) \times 180°$, and asked him, "Why does that property work?" To justify this rule, Luke gave an explanation for the angle sum of a pentagon. He carefully traced around a pentagon block, completed the picture shown in **figure 3,** and stated that the sum of the central angles was 360 degrees. He had learned while studying with Hogben that the sum of angles about a point is 360 degrees. Then he found that

Luke's first picture to demonstrate the formula

$360° \div 5 = 72°$, which he recorded as the measure of each central angle. Next he decided that the base angles of each of the five triangles were congruent. Luke applied a known Euclidean notion, that the angle sum for a triangle is 180 degrees, to ground his subsequent deductive argument that the sum of the two base angles in each triangle was 108 degrees, or $180° - 72° = 108°$. Next he concluded that because each of the base angles of the five triangles was congruent, the measure of an angle of the pentagon would be equal to the sum of the measures of any two base angles from the triangles. That sum also equaled 108 degrees. Then he found that the sum of the interior angles of the pentagon is 180×5, or 540, degrees. He did not offer a generalization of this result for all pentagons. Perhaps an appropriate question would have stimulated him to give a level 2 justification for the generalization.

As Luke talked through, and wrote about, the work shown in **figure 3,** he made a generalization that was an example of level 2 thinking: "The angle sum of *any* polygon would be equal to 360 degrees less [than 180 degrees times the number of sides of the polygon]." Luke's thinking can be characterized by a formula: $(n \times 180°) - 360°$, where n is the number of sides of the polygon. Note that Luke did not give this exact formula; it is offered to clarify his words. After Luke made his comment, his eyes lit up. He realized that he had not explained the formula he originally recalled: $(n - 2) \times 180°$.
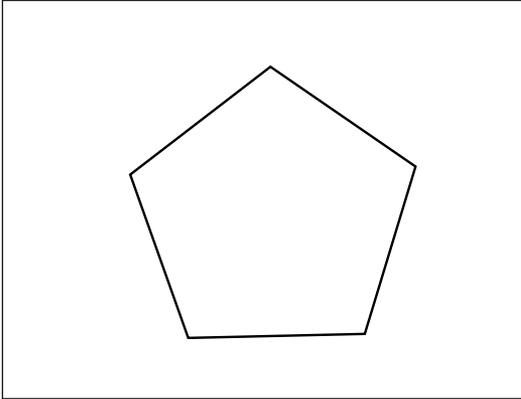
Because the two formulas are algebraically

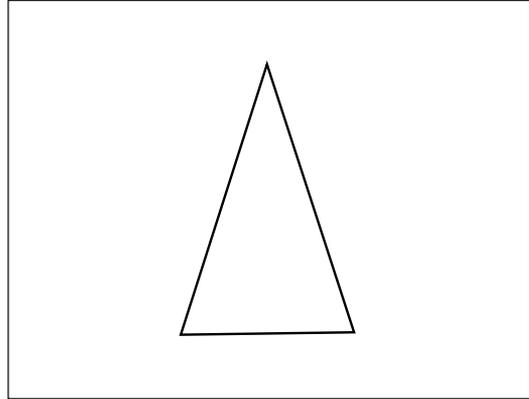**Pentablock Activity**

Find the measures of the interior angles of each polygon without using a protractor.
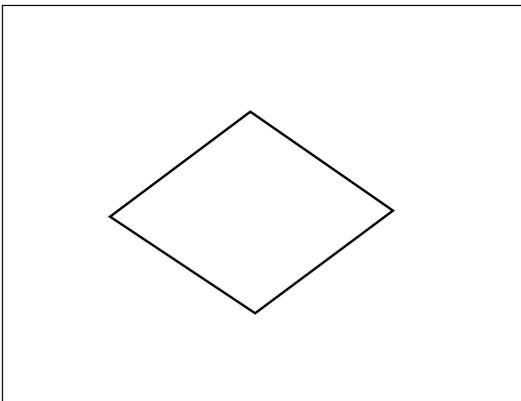
Name of yellow shape _____

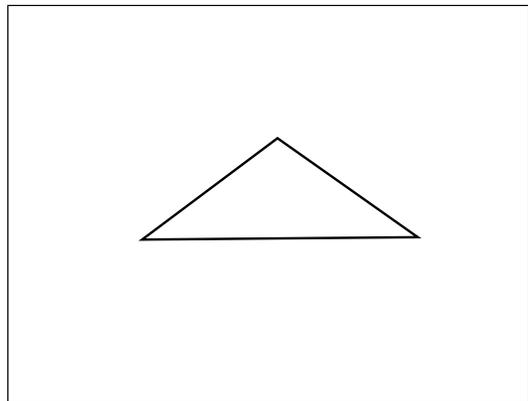Name of white shape _____

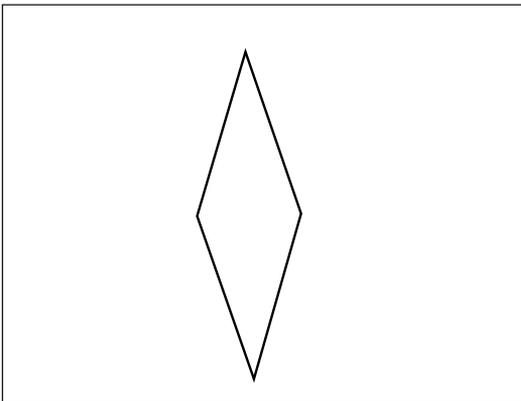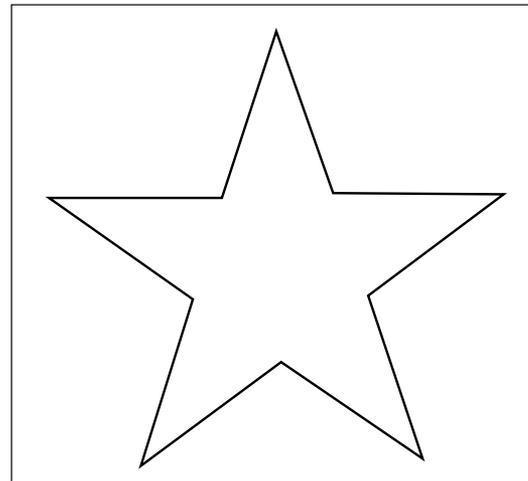Name of green shape _____

Name of purple shape _____

Name of pink shape _____

Name of black shape _____

(Adapted from Berman, Plummer, and Scheuer [1998])

From *Teaching Children Mathematics*, March 2001

(a) Luke's second picture to demonstrate the formula and (b) the Pentablocks

(a)

(b)
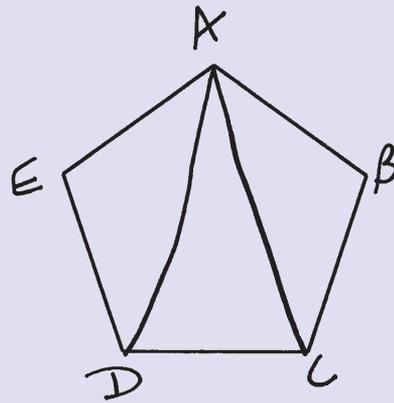
Luke's nine-sided polygon
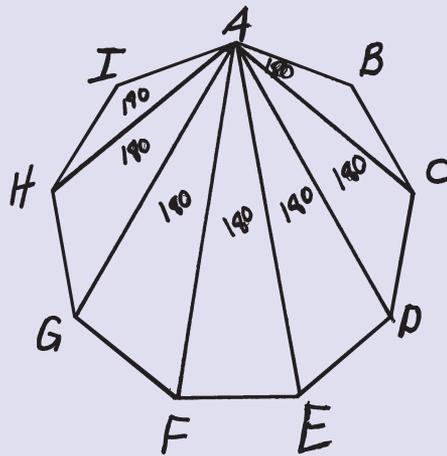
equivalent, they give the same result. The explanations for the formulas, however, are significantly different because each equation requires different procedures and different figures to accompany those explanations. Luke's attention to the mismatch between the explanation describing his work with the pentagon in **figure 3** and his formula, $(n − 2) × 180°$, is evidence of level 2 thinking. Luke's perceptive observation that his explanation did not match his rule exemplifies the characteristics of a mathematician.

Undaunted by the lack of fit between the formula and his drawing, Luke drew another picture, shown in **figure 4.** He placed the three appropriate triangle Pentablocks in the corresponding spaces of the pentagon shape in his drawing and wrote $180 × 3 = 540$. He stated that the polygon would always have two fewer triangles than the number

of its sides. Luke drew and referred to a regular nine-sided polygon, shown in **figure 5** (which is not in the Pentablock set), to verify his claim. Luke explained:

> I did a nine-gon. This is how I did it: $9 − 2 = 7$, 7 times 180 is 1260; 1260 divided by 9 is 140. So when I made my nine-gon, I measured each angle to be 140 degrees and it worked out perfectly. You can use this theory with any shape.

Luke then returned to his explanation of the formula that he had recalled at the beginning of the assessment:

> To do the thing to find the total of the . . . angles, there is a reason for having to subtract 2. It happens when you do the triangle way to find it. [See **fig. 4.**] There are the same number of vertices as sides, so I'll talk about vertices. There is one [vertex] that you start from. Then there are the two that are next to it (that can't be used). This would mean that you need to subtract 3, but when you make the last line, you make two triangles, so you only need to subtract 2.

Here, Luke demonstrated correct if-then reasoning, further demonstrating his level 2 thinking when he concluded, "So you only need to subtract 2." Luke continued, using his nine-sided polygon to demonstrate:

> You can only divide the [angle sum] total to find the angle of each side if the shape is regular. You can find the total of the . . . angles of an irregular polygon, but all the angles probably won't be the same.

Luke paid careful attention to whether his statements could be generalized. He correctly did not try to find the measure of each angle of irregular shapes by dividing the angle sum by the number of sides. This recognition of what notions can be generalized and when those generalizations apply is clearly level 2 thinking.

In general, Luke seemed to move easily between thinking with diagrams and recording with symbols. He distinguished between the two methods as verification strategies. Rather than draw $n - 2$ triangles on a nonconvex figure, Luke opted instead to accept the known formula: "[Drawing] the triangle thing won't work with a nonconvex figure, but the side-minus-2-times-180 will." He knew the shortcomings of using a diagram for showing the $n - 2$ triangles on nonconvex polygons. He completely abandoned the first diagram **(fig. 3),** in which he drew n triangles meeting at a central point, because it did not match his explanation for the formula $(n - 2) \times 180°$.

Luke explained that his $n - 2$ triangle diagram would not work for a nonconvex figure, because when all vertices are joined to a given vertex, some of the segments fall outside the shape. Luke pointed to the two vertices of the decagon on **sheet 1,** which, when joined, would produce a line segment falling outside the shape. He was reluctant to record this effort on his drawing. He had already stated that drawing the segments would not work, and he appeared unwilling to spend any more time with the task. He was confident in his use of the formula and clearly understood the difference between an example of, and an explanation for, the formula. Perhaps the ability to recognize that examples are insufficient as explanations is one of Luke's greatest skills as a gifted young mathematician.

After explaining his work with the formula, Luke returned to the task of finding the angle measurement of the remaining shapes on **sheet 1.** His work with the familiar quadrilaterals and triangles was fairly uneventful. For the green rhombus, he returned to the pentagon block. He found that each of the pentagon's angles was congruent to the larger angle of the rhombus; therefore, he recorded 108 degrees for that angle and the opposite angle. Then he subtracted $360° - 216° = 144°$ and recorded 72 degrees for the other two angles. To arrive at his answers for the smaller angle measurement, Luke used the ideas that *(a)* the angle sum for a quadrilateral was 360 degrees and *(b)* in a parallelogram, opposite angles are congruent. He knew these facts from his previous work with Hogben. His fluent use of these two ideas to deduce the angle measurements of the two smaller angles again represents level 2 thinking.

Luke continued using known angles, along with the facts that the angle sum of a triangle is 180 degrees and that the angle sum of a quadrilateral is 360 degrees, to find the remaining angle measurements for the pink rhombus and the two triangles. Hoiberg asked, "Why is the angle sum for a quadrilateral 360 degrees?" Luke's some-

Luke explains the angle sum of a quadrilateral.
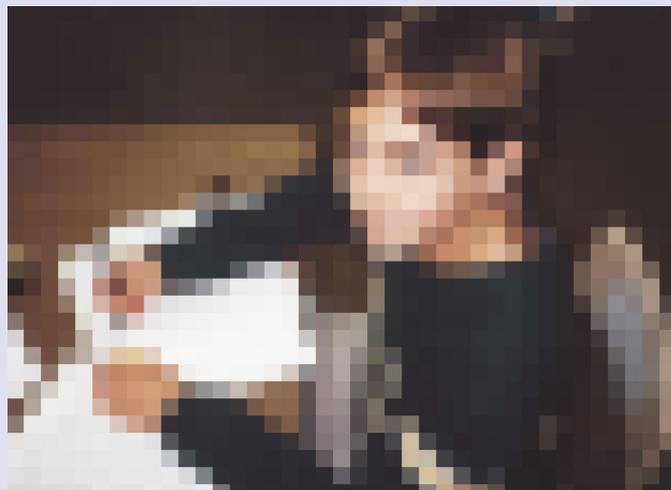
what amused facial expression (see **fig. 6**) accompanied his answer: "There are two triangles inside a quadrilateral, so there are two times 180 degrees inside a quadrilateral."

When he encountered the nonconvex decagon (see **sheet 1**), Luke was momentarily challenged. Because he had already stated that he did not want to draw his "triangle thing" on a nonconvex figure, he used another approach. He stacked blocks to show that the acute angle of the decagon was congruent to the acute angle of the pink rhombus, which he had found to measure 36 degrees. He then said that he knew the decagon's angle was also 36 degrees. He tried to determine the angle measurement of the decagon's larger interior angle by using symbolic numeral manipulation; he subtracted $180° - 36° = 144°$, but he could not decide

Luke's strategy to determine the exterior angle of the decagon

Name of black shape

$360° - 180°$ $180°$
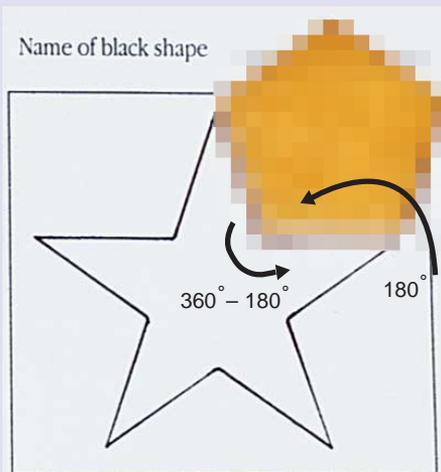
what to do with the result. Eventually, he placed the pentagon block flush against the exterior angle of the decagon, as shown in **fig. 7.** When Luke's symbolic manipulation was not meaningful to him, he readily used the blocks to focus his thinking or to verify his measurements or claims of congruence.

Luke then calculated that the large interior angle was 360° – 108°, or 252°. His use of the physical blocks to measure the angles marked Luke's voluntary return to level 1 activity when he needed to verify his claims before he could move forward. The van Heile theory claims that a learner must experience visual recording (level 0) before he or she is able to formulate descriptions at level 1 or engage in informal deduction at level 2 in completing geometric tasks. Luke's thinking processes outlined in this article seem to support this theory.

Teacher and Student: Partners in Learning

Hoiberg is locally and nationally recognized for her excellence in teaching mathematics and for regularly providing challenges for all her fifth-grade students. In Luke, she recognized an excellent learner. She worked diligently to give him opportunities that would challenge him to higher levels of thinking and push him toward becoming a mathematician. At times, Hoiberg worried about her ability to present Luke with situations or ask him questions that would lead to deeper mathematical thinking. Her concerns are elaborated in this description:

Before he entered my classroom, I knew Luke. I knew of Luke's family. His sister had been one of my students. I knew he was extremely bright. I worried about how I would challenge him. I needn't have worried. Luke loves learning and makes challenges for himself.

Later in the school year, Luke's teacher recognized an unusual opportunity to further nurture Luke's ability to think at higher levels.

I found another challenge when I signed up for a graduate class in geometry learning and teaching at the local university. I asked Luke if he'd like to look over my work with me. He loved it! He often helped me with assignments. He loved being my teacher! When it fit into his busy schedule, he came to class with me. He enjoyed sharing his theories with my fellow classmates.

One problem presented in the graduate class was to use the computer program The Geometer's Sketchpad (1995) to find the areas of successive midpoint quadrilaterals. **Figure 8** shows an example of this process. When several teachers in the class struggled with the problem, Luke clarified the wording of the problem, then solved it. He found that each of the successive midpoint quadrilaterals would be parallelograms and used the area tool to consider the areas of successive midpoint quadrilaterals. He summarized the investigation by observing, "I think that the area is cut in half each time that you make another polygon inside the old one. It's like shrinking by the powers of 2."

Luke's empirical discovery of this area relationship is an example of his level 1 thinking about the new topic. He did not try to explain why his observation was true (in part, because he was late for soccer practice). If we had examined simpler shapes, such as squares or rectangles, Luke might have been able to develop an informal explanation for why the area of the midpoint figure is half the area of the original (Welchman 2000).

Hoiberg was willing to risk allowing Luke to be her teacher, and Luke responded to the challenge. Conventional wisdom says that the best mathematics teachers are those who can push their students to gain knowledge beyond even that which is held by the teacher. This statement would seem particularly pertinent to teachers who work with gifted students. Luke's teacher pushed Luke toward the highest van Hiele levels, even though she was achieving some of those levels at the same time.

Luke's teacher used the van Hiele learning theory to pose questions and to interpret Luke's answers. She used this information to provide Luke with meaningful learning activities. Recognizing lower-level answers, she continued to ask questions that pushed Luke toward higher-level reasoning. She asked him to work alongside her as she completed her geometry assignments, which included writing proofs. In turn, Luke offered sound reasoning about his geometric ideas and

FIGURE 8

An example of the midpoint-quadrilateral problem
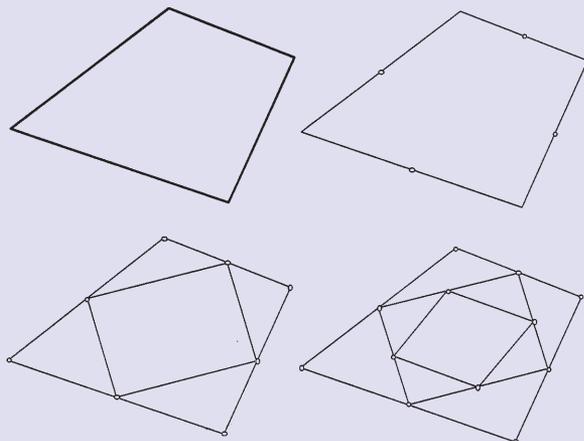
taught his teacher about geometry; he also showed her how to animate the mathematician waiting inside a gifted child.

Implications for Teaching

Although we have focused on using the van Hiele theory to stimulate the mathematical development of a gifted fifth grader, we want to stress that the van Hiele theory can be used with students of all ages and all ability levels. The kinds of tasks and questioning techniques used in assessing Luke can be incorporated into instruction to foster students' advancement through the levels of thinking. The task of finding the measurements of the interior angles of the Pentablocks releases the learner from reliance on protractors. In turn, the task gives the teacher deeper insight into the learner's reasoning about angle measurements, and this insight can be used to plan subsequent instruction.

The van Hiele theory provides the teacher with a tool to recognize the level of a student's thinking and to facilitate a student's advancement to successive levels by asking questions at the next higher level. For instance, when Luke said that he knew the interior angles of the pentagon were congruent because "they all look the same," the teacher prodded Luke forward. She continued to ask questions that encouraged Luke to use level 1, descriptive, thinking. She used his words or numbers to pose her questions, for example, "How did you know this angle was 108 degrees?" or "Why did you subtract 216 from 360?" At times, she asked him to explain how he knew that mathematics allowed him to make certain claims. At other times, she urged him to make generalizations or asked him to develop counterexamples: "Does this work for any polygon? What about an *n*-sided polygon? What about a nonconvex polygon?"

Encouraging children to answer questions at the higher levels of the van Hiele model requires careful thought. While some classmates are describing properties of shapes, using level 1 thinking, gifted children might be pressed to classify a set of triangles on the basis of these properties or to deduce ideas and relationships about the triangles. These children may be further pressed to develop their own definitions of a triangle or to deduce relationships between a triangle and a quadrilateral.

Teachers must select geometrically worthwhile tasks and problems, such as those on **sheet 1,** that have the potential to foster higher levels of thinking. Teachers can develop collections of worthwhile tasks for exceptional children, perhaps using them for special individual or small-group explorations. Another possibility is to assemble a learning center that includes some basic tasks, along with challenges. Resources for center activities include *Geometry and Spatial Sense* (Leiva 1993); *Geometry in the Middle Grade*s (Geddes 1992); *Shapes Alive!* (Leeson 1993); and *Geometry and Geometric Thinking,* a focus issue of *Teaching Children Mathematics* (NCTM 1999). From the moment that each child enters a classroom, the teacher should begin listening for, and recording clues to, the child's van Hiele level. The teacher can then design geometry tasks to match the student's level and eventually raise the child's thinking to a higher level.

Working with an exceptional child such as Luke can make a teacher aware of the special mathematical potential in certain children and can help the teacher observe more carefully the mathematical potential in all students. Readers are encouraged to interview several of their students to determine their van Hiele levels of thinking about a certain concept, then to use the students' answers to design and develop questions at or above those levels.

References

Berman, Sheldon, Gary A. Plummer, and Don Scheuer. *Investigating Mathematics with Pentablocks.* White Plains, N.Y.: Cuisenaire Co. of America, 1998.

Fuys, David, Dorothy Geddes, and Rosamond Tischler. *The van Hiele Model of Thinking in Geometry among Adolescents. Journal for Research in Mathematics Education* Monograph Series No. 3. Reston, Va.: National Council of Teachers of Mathematics, 1988.

Geddes, Dorothy. *Geometry in the Middle Grades. Curriculum and Evaluation Standards* Addenda Series, Grades 5–8. Reston, Va.: National Council of Teachers of Mathematics, 1992.

The Geometer's Sketchpad. Berkeley, Calif.: Key Curriculum Press, 1995. Software.

Leeson, Neville. *Shapes Alive!* Lincolnshire, Ill.: Learning Resources, 1993.

Leiva, Miriam A., ed. *Geometry and Spatial Sense. Curriculum and Evaluation Standards* Addenda Series, Grades K–6. Reston, Va.: National Council of Teachers of Mathematics, 1993.

National Council of Teachers of Mathematics. "Geometry and Geometric Thinking" (Focus Issue). *Teaching Children Mathematics* 5 (February 1999).

van Hiele, Pierre. *Structure and Insight: A Theory of Mathematics Education.* Orlando, Fla.: Academic Press, 1986.

Welchman, Rosamond. "Investigations: Midpoint Shapes." *Teaching Children Mathematics* 6 (April 2000): 506–9.