



Guidance

Curriculum & Standards

# Key Stage 3 National Strategy

## What is a fraction?

Teachers of  
mathematics

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# **Key Stage 3**

## *National Strategy*

What is a fraction?

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# Preface

This self-study booklet is on the topic of fractions, ratio and proportion.

It was originally written to support and develop the subject knowledge of teachers in Key Stages 2 and 3. The reasons for wanting to do this are:

- The 2000 National Curriculum gives greater emphasis to this topic, particularly ratio and proportion. This is developed in the Primary and Key Stage 3 Frameworks for teaching mathematics.
- Ratio and proportion is a mathematical topic which less experienced teachers frequently say they find most difficult to teach.
- There are many interconnections which need to be understood by teachers if they are to present the topic in a coherent way.

The content of the booklet addresses mathematical content up to National Curriculum levels 7 and 8. It is being provided as background reading for Key Stage 3 mathematics teachers. The reasons for this are:

- to foster a systematic approach to teaching the topic, with improved progression from Key Stage 2 to Key Stage 3, particularly in the steps from informal to more formal methods;
- to encourage the development of explicit links in mathematics (and to other subjects), particularly proportional thinking as applied in number, algebra, shape and space, and handling data.

As you read through the booklet, select one or two questions from each exercise that look interesting. As you do so, consider the following questions, related to the teaching of this topic in your school:

- Do we teach fractions, ratio and proportion in ways that make these concepts, and the links between them, clear to pupils?
- Do we ensure that pupils are taught methods for solving problems and that they do not just rely on informal approaches of their own devising?
- Do we make explicit to pupils the links between proportional thinking in number, algebra, shape and space, and handling data?

# Introduction

## Fractions and division

This booklet discusses the nature of fractions and the related ideas of ratio and proportion. It aims to clarify the meanings of these terms and the links between them. Each main section concludes with a short exercise, to test your understanding and application of the ideas. There are also a few more searching questions for you to think about.

To gain a clear understanding of fractions, it is worth noting these points:

- The number system evolved initially for the purpose of counting, using positive whole numbers. Later, it was extended to include zero and negative numbers (integers) and fractions (or rational numbers). Then other types of number emerged: irrational, transcendental and complex numbers are beyond the scope of this booklet, but they are all useful in mathematics and its applications. A good mathematical dictionary will define these numbers for you if you do not know what they are.
- The notion that quantities can be subdivided into equal parts is within the experience of young children – for example, they know about halving things. Fractions are numbers which arise from the operation of division, in two ways. A fraction can be thought of as:
  - the result of dividing a whole number into equal parts and taking one or more of those parts (the *part/whole* aspect), or
  - how many times one number can be divided by another (the *ratio* aspect).

The introduction of negative numbers enables counting back to continue indefinitely and makes it possible to give an answer to subtractions such as  $2 - 5$ . The operation of multiplication stays within the set of whole numbers, but division does not. For example, calculations such as  $8 \div 2$  or  $15 \div 3$  have whole number answers but  $1 \div 2$  and  $12 \div 7$  do not. To include the results of these calculations in the number system a new type of number is required, namely a *fraction*:

*A fraction results from the division of one whole number by another.*

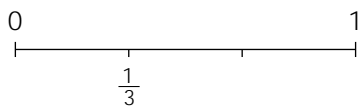
If  $p$  and  $q$  are whole numbers, then  $p \div q$  is called a *fraction* or, in more technical language, a *rational number*. It is usually written as  $\frac{p}{q}$ , the fraction bar or line being equivalent to the division sign:  $p \div q = \frac{p}{q}$ .

Sometimes fractions reduce to a whole number (e.g.  $\frac{8}{2} = 4$ ,  $\frac{15}{3} = 5$ ). Sometimes they do not (e.g.  $\frac{1}{2}$ ,  $\frac{12}{7}$ ). Important fractions to consider are divisions of 1, because they give the familiar fraction names:  $\frac{1}{2}$  (one half),  $\frac{1}{3}$  (one third),  $\frac{1}{4}$  (one quarter),  $\frac{1}{5}$  (one fifth), ...

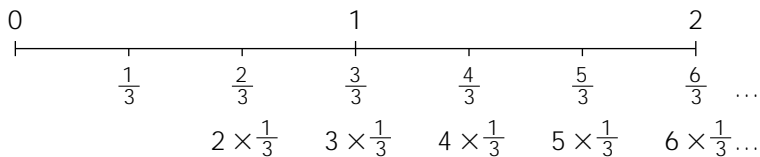
# Chapter 1

## Fractions as divisions of a whole number into equal parts

Interpreting division as sharing, a fraction such as  $\frac{18}{6}$  can be read 'If 18 is divided into 6 equal parts, what is the number in each part?' Similarly,  $\frac{1}{3}$  can be thought of as the result of dividing 1 into 3 equal parts and taking one of those parts. This can be illustrated on the number line:



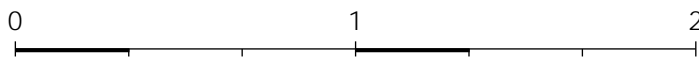
Multiples of  $\frac{1}{3}$ , also called thirds (two thirds, three thirds, four thirds, ...), are marked at equal spaces on the number line:



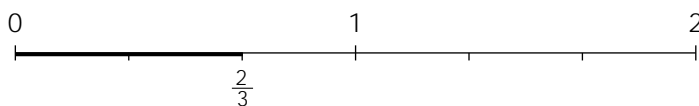
Because thirds are represented by line segments of equal lengths, the diagram also illustrates how they can be added and subtracted. For example,  $\frac{2}{3} + \frac{3}{3} = \frac{5}{3}$  and  $\frac{5}{3} - \frac{3}{3} = \frac{2}{3}$ .

So far, a fraction such as  $\frac{2}{3}$  has been shown as 2 lots of  $\frac{1}{3}$  ( $2 \times \frac{1}{3}$ ). Returning to the definition given on page 4, it can also be thought of as  $2 \div 3$ . To illustrate this, first divide 2 into three equal parts by taking  $\frac{1}{3}$  from each unit.

( $2 \times \frac{1}{3}$ ):



Then put them together ( $2 \div 3$ ):



Another way of expressing this result is to say that  $\frac{2}{3}$  is  $\frac{1}{3}$  of 2 ( $\frac{1}{3} \times 2$ ), thinking of  $\frac{1}{3}$  as an operator. So there are three ways of interpreting the fraction  $\frac{2}{3}$ :

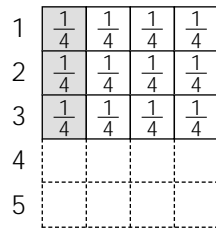
$$2 \div 3 \quad \frac{1}{3} \text{ of } 2 \left( \frac{1}{3} \times 2 \right) \quad 2 \text{ lots of } \frac{1}{3} \left( 2 \times \frac{1}{3} \right)$$

It is important to recognise that  $\frac{1}{3}$  of 2 and 2 lots of  $\frac{1}{3}$  lead to the same answer, so that the commutative law applies:  $\frac{1}{3} \times 2 = 2 \times \frac{1}{3}$ .

Traditionally, fractions are also represented by areas. For example,  $\frac{3}{4}$  may be shown as 3 lots of  $\frac{1}{4}$  ( $3 \times \frac{1}{4}$ ) by subdividing a unit square:



However, the diagram is limited to this particular interpretation of  $\frac{3}{4}$ . To represent the meaning of  $\frac{3}{4}$  as  $3 \div 4$  or  $\frac{1}{4}$  of 3 ( $\frac{1}{4} \times 3$ ), a different diagram is required. Start by drawing a rectangle to represent 1, then two more rectangles of the same size underneath, to represent 2 and 3. Finally, divide the rectangles into quarters by drawing three vertical lines:



This diagram shows all the relationships that need to be illustrated, retaining the link with division and avoiding the false impression that a fraction is part of 1 and never of any other number. The dotted lines indicate how the diagram is extended to show  $\frac{1}{4}$  of 4,  $\frac{1}{4}$  of 5, and so on.

The language of fractions can be reinforced using fraction tables, constructed in two ways:

1 times $\frac{1}{4} = \frac{1}{4}$	$\frac{1}{4}$ of 1 = $\frac{1}{4}$
2 times $\frac{1}{4} = \frac{1}{2}$ (or $\frac{2}{4}$ )	$\frac{1}{4}$ of 2 = $\frac{1}{2}$ (or $\frac{2}{4}$ )
3 times $\frac{1}{4} = \frac{3}{4}$	$\frac{1}{4}$ of 3 = $\frac{3}{4}$
4 times $\frac{1}{4} = 1$ (or $\frac{4}{4}$ )	$\frac{1}{4}$ of 4 = 1 (or $\frac{4}{4}$ )
5 times $\frac{1}{4} = 1\frac{1}{4}$ (or $\frac{5}{4}$ )	$\frac{1}{4}$ of 5 = $1\frac{1}{4}$
6 times $\frac{1}{4} = 1\frac{1}{2}$ (or $\frac{6}{4}$ )	$\frac{1}{4}$ of 6 = $1\frac{1}{2}$
7 times $\frac{1}{4} = 1\frac{3}{4}$ (or $\frac{7}{4}$ )	$\frac{1}{4}$ of 7 = $1\frac{3}{4}$
8 times $\frac{1}{4} = 2$ (or $\frac{8}{4}$ )....	$\frac{1}{4}$ of 8 = 2...

The last diagram can also be easily adapted to illustrate the addition and subtraction of fractions with the same denominator, e.g.  $\frac{3}{4} + \frac{2}{4} = \frac{5}{4} = 1\frac{1}{4}$ :



To summarise:

- Fractions are numbers of the form  $\frac{p}{q}$ , where  $p$  and  $q$  are whole numbers.
- $\frac{p}{q}$  means  $p \div q$ .
- $\frac{p}{q}$  can also be interpreted as  $p \times \frac{1}{q}$  ( $p$  lots of  $\frac{1}{q}$ ) or as  $\frac{1}{q} \times p$  (one  $q$ th of  $p$ ).
- $q$  is called the *denominator*, because it gives the name of the fraction.
- $p$  is called the *numerator*, because it gives either the number which is divided ( $p \div q$  or  $\frac{1}{q} \times p$ ) or the number of fractional parts ( $p \times \frac{1}{q}$ ).
- Fractions less than 1 have  $p < q$  and are called *proper fractions*.
- Fractions equal to 1 have  $p = q$  ( $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \dots$ ).
- Fractions greater than 1 have  $p > q$  and are called *improper fractions*.
- Improper fractions can also be written as *mixed numbers*, that is, a whole number with a proper fraction. For example:  $\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1\frac{2}{3}$ .

### Exercise A

1. What is another way of writing  $1 \div 4$ ?
2. Which is bigger,  $\frac{1}{6}$  or  $\frac{1}{7}$ ? Why?
3. Name a fraction that lies between  $\frac{8}{7}$  and  $\frac{11}{7}$ . Name another, and another...
4. Write the answer to  $12 \div 7$  as an improper fraction and as a mixed number.
5. Write the answer to  $\frac{2}{5} + \frac{7}{5}$  as a mixed number.
6. Draw a diagram to explain why  $\frac{3}{5}$  (i.e.  $3 \div 5$  or  $\frac{1}{5}$  of 3) = 3 lots of  $\frac{1}{5}$ .
7. We ate  $\frac{1}{3}$  of a cake today. What fraction is left for tomorrow?
8. I spent half my pocket money on a book, then half of the remainder on fish and chips. What fraction of my money is left?

And here is something to think about:

*Fractions can be thought of as labels for points on the number line, between the whole numbers. If all imaginable fractions are marked on the line, will they fill it completely, or will there still be gaps?*

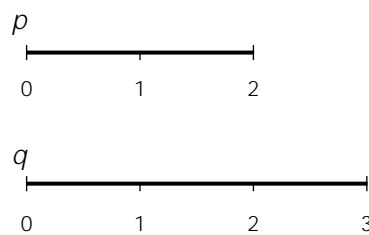


# Chapter 2

## Fractions as ratios – how many times one number can be divided by another

Chapter 1 focused on division as sharing. The other way of interpreting division is as grouping. A fraction, when viewed in this way, is called a *ratio* – *the number of times one number can be divided by another*. So the fraction  $\frac{18}{6}$  can be interpreted as ‘How many times can 18 be divided by 6?’ or ‘What does 6 have to be multiplied by to make 18?’. In this case the answer is a whole number (3) but this is not always the case. For example,  $\frac{2}{3}$  can be thought of as the number of times 2 can be divided by 3 ( $\frac{2}{3}$  of a time) and, inversely,  $\frac{3}{2}$  can be thought of as the number of times 3 can be divided by 2 ( $\frac{3}{2}$  or  $1\frac{1}{2}$  times).

It is useful to consider the divisions  $\frac{2}{3}$  and  $\frac{3}{2}$  together. Moving away from the part/whole aspect, the comparison of one whole number with another can be illustrated by two line segments, of length  $p$  and  $q$ , say. (If you wanted to, you could make a concrete form for these representations with Cuisenaire rods, linking cubes, or strips of card.)



There are several equivalent ways in which these relationships can be expressed:

- ratio of  $p$  to  $q = 2 : 3$       ratio of  $q$  to  $p = 3 : 2$
- $\frac{p}{q} = \frac{2}{3}$        $\frac{q}{p} = \frac{3}{2}$
- $p = \frac{2}{3} \times q$        $q = \frac{3}{2} \times p$

In the last two equations, fractions are used as operators (multipliers). This requires some explanation. Think first about the meaning of  $\frac{p}{q}$ . ‘What is  $p$  divided by  $q$ ?’ is equivalent to ‘What does  $q$  have to be multiplied by to give  $p$ ?’. In this example, the answer is  $\frac{2}{3}$ . So the equation  $p = \frac{2}{3} \times q$  can be read ‘ $p$  equals two thirds of  $q$ ’. The operator ‘ $\frac{2}{3} \times$ ’ means ‘ $\frac{2}{3}$  of’ or ‘2 lots of  $\frac{1}{3}$  of’. It is equivalent to dividing by 3 and multiplying by 2.

Similarly, ‘What is  $q$  divided by  $p$ ?’ is equivalent to ‘What does  $p$  have to be multiplied by to give  $q$ ?’. In this example, the answer is  $\frac{3}{2}$ . So the equation  $q = \frac{3}{2} \times p$  can be read ‘ $q$  equals three halves of  $p$ ’. The operator ‘ $\frac{3}{2} \times$ ’ means ‘ $\frac{3}{2}$  of’ or ‘3 lots of  $\frac{1}{2}$  of’. It is equivalent to dividing by 2 and multiplying by 3.

There are two further points to note, relating to terminology:

- The term *proportion* is often used in the everyday sense of a fractional part.
- Inverted fractions are called *reciprocals*. For example,  $\frac{3}{2}$  is the reciprocal of  $\frac{2}{3}$  and  $\frac{2}{3}$  is the reciprocal of  $\frac{3}{2}$ .

To divide a number into parts in a given ratio, first find the total number of parts involved. For example, to divide 30 in the ratio of 2 : 3, there will be 5 parts altogether (2 + 3). Each part will be  $\frac{1}{5}$  of 30 = 6. So the required numbers are 12 ( $\frac{2}{5} \times 30$ ) and 18 ( $\frac{3}{5} \times 30$ ). Generalising, if  $n$  is to be divided into two parts in the ratio  $p : q$ , the parts will be  $p/(p + q)$  of  $n$  and  $q/(p + q)$  of  $n$ .

**To summarise:**

- Ratios are fractions which express how many times one number can be divided by another.
- The ratio  $p : q$  corresponds to the fraction  $\frac{p}{q}$ .
- The *inverse* ratio  $q : p$  corresponds to the *reciprocal* fraction  $\frac{q}{p}$ .
- Using the relationship between multiplication and division, one number can be expressed as a fraction of another:  $p = \frac{p}{q} \times q$  and the *inverse* relationship  $q = \frac{q}{p} \times p$ .
- Although it is common to refer to one number as a *proportion* of another, note that 'proportion' has a broader technical meaning in mathematics (of which more later).
- The operation ' $\frac{1}{q} \times$ ' (one  $q$ th of) is equivalent to dividing by  $q$  (e.g. halving ' $\frac{1}{2} \times$ ' is equivalent to dividing by 2).
- The operation ' $\frac{p}{q} \times$ ' ( $p$   $q$ ths of) is equivalent to dividing by  $q$  and multiplying by  $p$ , and its inverse is ' $\frac{q}{p} \times$ ' ( $q$   $p$ ths of), which is equivalent to dividing by  $p$  and multiplying by  $q$ .
- $n$  divided in the ratio  $p : q$  will have parts  $p/(p + q)$  of  $n$  and  $q/(p + q)$  of  $n$ .

**Exercise B**

1. 'The width of the Millennium Dome is roughly 6 times the wingspan of the Angel of the North.' Express this statement in two other ways, using the language of fractions and ratio.
2. Express (a) 5 as a fraction of 8 and (b) 8 as a fraction of 5.
3. I started with £20 and spent £17. What proportion of my money is left?
4. £36 is divided into £4.50 amounts. How many people can share the money?
5. £160 is to be divided between two people in the ratio 3 : 5. How much does each person get?
6. If  $p$  is  $\frac{3}{5}$  of  $q$ , what fraction is  $q$  of  $p$ ?
7. A car fuel tank holds 45 litres.  $\frac{2}{5}$  of the fuel has been consumed. How many litres are left?
8. The salaries of myself and of my friend have both increased by  $\frac{1}{4}$  over a period of 5 years.
  - (a) What fraction is my current salary of my salary 5 years ago? If my salary 5 years ago was £20 000, what is my current salary?
  - (b) What fraction was my friend's salary 5 years ago, of her current salary? If her salary is now £30 000, what was it 5 years ago?

And here is something to think about:

*Multiplication where one of the numbers is a fraction has been discussed. How could multiplication of a fraction by another fraction be explained? How could division of a whole number by a fraction, or division of a fraction by a fraction, be explained?*

# Chapter 3

## Equivalent fractions

Points on the number line can have more than one label. For example, counting in halves, the point 1 would be labelled  $\frac{2}{2}$ , counting in thirds it would be  $\frac{3}{3}$ , and so on. The set of fractions  $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \dots$  are said to be *equivalent* and any one can replace any of the others.  $\frac{1}{1}$  (or just 1) involves the smallest numbers and is regarded as the simplest form.

A set of equivalent fractions involves a constant multiplier relating numerator to denominator. For example:

$$\blacksquare \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5} = \dots \quad (\text{numerator} = 2 \times \text{denominator})$$

$$\blacksquare \frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \dots \quad (\text{numerator} = \frac{1}{2} \times \text{denominator})$$

Sets of equivalent fractions are formed by multiplying or dividing numerator and denominator by the same number:

$$\frac{p}{q} = \frac{a \times p}{a \times q} = \frac{p \div a}{q \div a}$$

The process of dividing numerator and denominator by a common factor is called *cancelling*. Replacing a fraction by an equivalent one, in which numerator and denominator are the smallest possible whole numbers, is called *simplifying*, or *reducing the fraction to its lowest terms*.

Every fraction is a member of two special families – a family of fractions with the same name (denominator) but different values, and a family of equivalent fractions with different denominators. For example,  $\frac{1}{3}$  is the *smallest* fraction in the set of thirds:  $\{\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \dots\}$ . It is also the *simplest* in the set of equivalent fractions:  $\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \dots\}$ .

Equivalence is used when ordering, adding and subtracting fractions. Consider the fractions  $\frac{5}{6}$  and  $\frac{7}{8}$ . Which is larger? There are several ways of finding an answer, but one method is to search the sets of equivalent fractions for those with the same denominator:

$$\left\{ \frac{5}{6}, \frac{10}{12}, \frac{15}{18}, \frac{20}{24}, \frac{25}{30}, \frac{30}{36}, \dots \right\}$$

$$\left\{ \frac{7}{8}, \frac{14}{16}, \frac{21}{24}, \frac{28}{32}, \frac{35}{40}, \frac{42}{48}, \dots \right\}$$

$$\text{Since } \frac{21}{24} > \frac{20}{24} \text{ it follows that } \frac{7}{8} > \frac{5}{6}.$$

Fractions can be added or subtracted only when they have the same denominator, rather as quantities can be added or subtracted only if they are measured in the same units (pounds cannot be added to euros, or feet to centimetres!). So, to complete calculations such as  $\frac{7}{8} + \frac{5}{6}$  or  $\frac{7}{8} - \frac{5}{6}$ , it is necessary to find a common name for the two fractions by searching for equivalents:  $\frac{21}{24} + \frac{20}{24} = \frac{41}{24}$  or  $\frac{21}{24} - \frac{20}{24} = \frac{1}{24}$ .

### Exercise C

1. Write three more fractions equivalent to  $\frac{3}{8}$ .
2. Write three more fractions equivalent to  $\frac{8}{6}$ .
3. Write the fraction  $\frac{32}{36}$  in its simplest form.
4. Which is bigger,  $\frac{4}{5}$  or  $\frac{5}{6}$ ? (Hint: Think of  $\frac{4}{5}$  as  $1 - \frac{1}{5}$  and  $\frac{5}{6}$  as  $1 - \frac{1}{6}$ )
5. Which is bigger,  $\frac{4}{7}$  or  $\frac{5}{9}$ ? (Hint: Find equivalent fractions)
6. What is  $\frac{4}{7} + \frac{5}{9}$ ?
7. What is the difference between  $\frac{4}{7}$  and  $\frac{5}{9}$ ?

# Chapter 4

## Proportion as equality of ratios

Consider the set of fractions (or ratios) equivalent to 5:

$$\frac{5}{1} = \frac{10}{2} = \frac{15}{3} = \frac{20}{4} = \frac{25}{5} = \dots$$

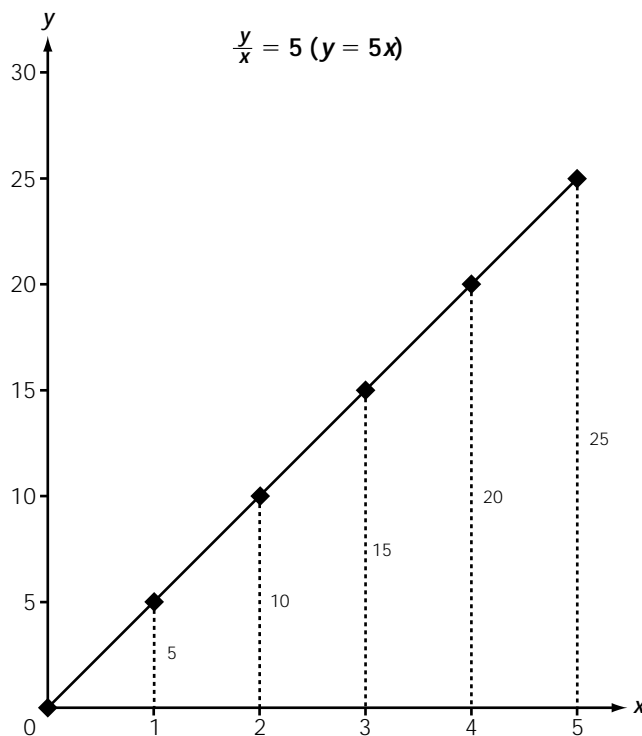
$$5 : 1 = 10 : 2 = 15 : 3 = 20 : 4 = 25 : 5 = \dots$$

Because of the equality of these ratios, the numerators {5, 10, 15, 20, 25, ...} and denominators {1, 2, 3, 4, 5, ...} are called *proportional sets* and the pairs of corresponding numbers (5, 1), (10, 2), (15, 3), (20, 4), (25, 5), ... are said to be *in proportion*.

*A proportion is an equality of ratios.*

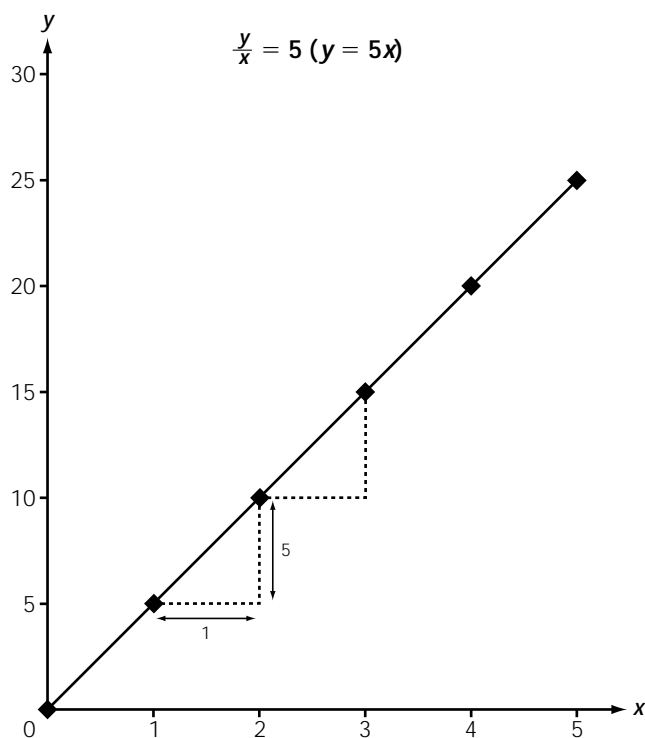
It is helpful to look at the relationship graphically. Representing the set {5, 10, 15, 20, 25, ...} by the variable  $y$  and the set {1, 2, 3, 4, 5, ...} by the variable  $x$ , then  $\frac{y}{x} = 5$  (or  $y = 5x$ ) for each corresponding number pair. The graph of  $y$  against  $x$  consists of a set of points in a straight line through the origin.

$x$	1	2	3	4	5	...
$y$	5	10	15	20	25	...



The vertical lines added to the diagram draw attention to the equal ratios  $\frac{5}{1}, \frac{10}{2}, \frac{15}{3}, \frac{20}{4}, \frac{25}{5}, \dots$ , represented by the sides of right-angled triangles, enlarging from the origin.

Another aspect to consider is how  $y$  changes with  $x$ . From the table, you can see that  $y$  changes by 5 for every 1 of  $x$ . *Rate* is the term used to describe how one variable changes with another. In this case we have an example of a *constant rate of change*, equal to 5. This can be illustrated by drawing different lines on the graph, as shown below.



The right-angled triangles, drawn stepwise from point to point, show a rise of 5 ( $y$ ) for every step of 1 ( $x$ ). Note that the phrases 'for every', 'in every', 'to every' or simply 'per', are all used when describing a constant rate (e.g. miles per hour).

Other sets of equal ratios also generate proportional sets. For example:

- $\frac{1}{5} = \frac{2}{10} = \frac{3}{15} = \frac{4}{20} = \frac{5}{25} = \dots$       $\frac{y}{x} = \frac{1}{5}$  or  $y = \frac{1}{5}x$
- $\frac{3}{2} = \frac{6}{4} = \frac{9}{6} = \frac{12}{8} = \frac{15}{10} = \dots$       $\frac{y}{x} = \frac{3}{2}$  or  $y = \frac{3}{2}x$

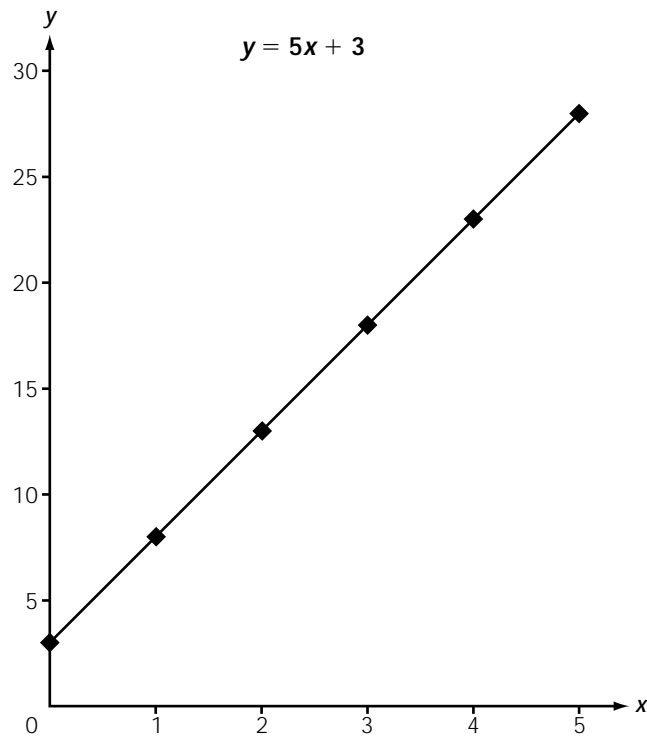
The graphs of these proportions are straight lines through the origin. The rate at which  $y$  changes with  $x$  is constant ( $\frac{1}{5}$  and  $\frac{3}{2}$  respectively).

Generalising, when  $\frac{y}{x} = m$  (or  $y = mx$ ), the constant of proportionality is  $m$  and the rate of change is  $m$  of  $y$  for every 1 of  $x$ . For different values of  $m$ , the graphs all go through the origin. The steepness or *gradient* of the line is measured by  $m$ .

But proportions are not the only relationships that involve a constant rate of change. Think about these sets of numbers:

$x$	1	2	3	4	5
$y$	8	13	18	23	28

The rate of change is still 5, but the ratios of successive pairs of values are not constant ( $\frac{8}{1} = 8$ ,  $\frac{13}{2} = 6\frac{1}{2}$ ,  $\frac{18}{3} = 6$ , and so on). The relationship in this case is  $y = 5x + 3$ . It is a *linear relationship*, because the graph is a straight line, but it does not go through the origin.



**To summarise:**

- A proportion is an equality of ratios.
- *Equivalent fractions* are formed from proportional sets of numbers, in which there is a constant multiplier relating numerators to denominators.
- Where two or more pairs of numbers represented by the variables  $(x, y)$  are such that  $\frac{y}{x} = m$ ,  $m$  being a constant number, the numbers are *in proportion* and  $m$  is called the *constant of proportionality* (sometimes thought of as a *rate*, sometimes as a *scale factor*):

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \dots = m$$

(Strictly speaking, this is called *direct proportion*, to distinguish it from *inverse proportion*.)

- The graph of  $y = mx$  consists of points lying in a straight line through the origin. If the variables are discrete, like the whole numbers in the examples given, the graph will be a set of discrete points. If the variables are continuous, the graph will be a continuous line.
- Thinking of  $x$  and  $y$  as variable numbers,  $m$  describes the *rate of change* of  $y$  with respect to  $x$ .  $y$  changes by  $m$  for every 1 of  $x$ .
- The phrases 'for every', 'in every' and 'to every', or simply 'per', are all used when describing a constant rate.
- Graphically, the rate of change is represented by  $\text{rise} \div \text{step}$ , and  $m$  measures the *gradient* of the graph.

### Exercise D

1. Which of these number pairs are in proportion and which are not?  
Convince a friend.

- (a) (12, 3), (20, 4)
- (b) (6, 48), (21, 168)
- (c) (5, 15), (10, 20)
- (d) (28, 7), (96, 24), (16, 4)

2. A photograph is 150 mm by 100 mm. Which of the following gives the dimensions of an accurate enlargement of the photograph?

- (a) 450 mm by 350 mm
- (b) 375 mm by 250 mm

What is the scale factor of the enlargement (i.e. the constant of proportionality)?

3. In each of these examples the relationship between  $y$  and  $x$  is linear, that is, as  $x$  changes,  $y$  changes at a constant rate. In each case, state the rate of change and determine whether  $y$  is proportional to  $x$ .

(a) $x$	1	2	3	4	5	...
$y$	6	10	14	18	22	...

(b) $x$	1	2	3	4	5	...
$y$	0.2	0.4	0.6	0.8	1	...

(c) $x$	1	2	3	4	5	...
$y$	12	10	8	6	4	...

(d) $x$	5	10	15	20	25	...
$y$	8	16	24	32	40	...

Plot graphs for each of these relationships and relate the features of your graphs to what you have discovered. For each of the examples, can you find an equation connecting  $y$  and  $x$ ?

4. Which of the following situations are examples of direct proportion and which are not?

- (a) Cost of purchasing a given quantity of tomatoes at £1.38 per kilogram.
- (b) Distance travelled on a bicycle in a given time, at a steady speed of 15 mph.
- (c) Number of dollars you can purchase for your money, at the rate of \$1.45 for every pound sterling, together with a flat rate commission of £5 per transaction.
- (d) Table of distances in miles converted into kilometres.



# Chapter 5

## Solving problems on ratio and proportion

Practical problems on ratio and proportion commonly involve two pairs of quantities in proportion, three of the quantities being known and the fourth having to be found. Usually the data have to be reformulated in some way, and often the calculation is completed in two stages. The nature of the problem and the particular numbers involved influence the choice of method:

- The *scaling method* involves finding a multiplier or *scale factor*. For example, a recipe for 4 people can be adapted for 8 using the scale factor 2, i.e. double the quantities.
- The *unitary method* involves finding a rate, the amount of one quantity per *unit* (or 1) of the other. For example, 9 miles walked in 3 hours converts to a rate of 3 mph, so the distance walked in 5 hours at this rate would be  $5 \times 3 = 15$  miles.

The scaling method is suitable for mental calculation if you can spot the required multiplier fairly quickly, or use a strategy like repeated doubling/halving:

In 15 minutes I travelled 18 km. If I travelled at the same speed, how far would I go in (a) 45 minutes, (b) 25 minutes?

(a) In 15 minutes I travelled 18 km.

In 45 minutes I would travel 54 km (noting  $\frac{45}{15} = 3$  and multiplying by 3).

(b) In 15 minutes I travelled 18 km.

In 5 minutes I would travel 6 km (times are multiples of 5 minutes, so divide by 3).

In 25 minutes I would travel 30 km (multiplying by 5).

The unitary method is suitable for mental calculation if the rate can be found by an easy division:

In 15 minutes I travelled 30 km. If I travelled at the same speed, how far would I go in 23 minutes?

In 15 minutes I travelled 30 km.

In 1 minute I would travel 2 km (dividing 30 by 15).

In 23 minutes I would travel 46 km (multiplying 2 by 23).

Here are three examples to illustrate both approaches. If you used a calculator, it would make either method suitable, whatever the particular numbers involved.

### Example 1

Peanuts cost 60p for 100 grams. How much would I pay for 350 grams?

#### **Scaling method**

Find the cost for 350 grams, given the cost for 100 grams.

350 grams is  $\frac{350}{100} = 3.5$  times as much as 100 grams.

So the cost will be  $3.5 \times 60\text{p}$ .

#### **Unitary method**

100 grams of peanuts cost 60p.

So 1 gram of peanuts would cost  $\frac{60}{100} \text{ p} = 0.6\text{p}$ .

So 350 grams would cost  $350 \times 0.6\text{p}$ .

### Example 2

Raisins cost 80p for 100 grams. I pay £2 for a bag of raisins. How many grams of raisins do I get?

#### **Scaling method**

Find the amount you can buy for 200p (£2), given how much you can buy for 80p.

200p is  $\frac{200}{80} = 2.5$  times as much money as 80p.

So the amount will be  $2.5 \times 100$  grams.

#### **Unitary method**

80p will buy 100 grams.

So 1p will buy  $\frac{100}{80} = 1.25$  grams.

So 200p (£2) will buy  $200 \times 1.25$  grams.

### Example 3

What is a distance of 300 miles converted into kilometres, given that 5 miles is approximately 8 kilometres?

#### **Scaling method**

Convert 300 miles to kilometres, given how many kilometres there are in 5 miles.

300 miles is  $\frac{300}{5} = 60$  times as far as 5 miles.

So the distance will be  $60 \times 8$  kilometres.

#### **Unitary method**

5 miles is approximately 8 kilometres.

So 1 mile is approximately  $\frac{8}{5} = 1.6$  kilometres.

So 300 miles is approximately  $300 \times 1.6$  kilometres.

When you solve problems, clarity of thinking and explanation go together. What relationship does the scaling represent? Which quantity is to be converted to a unitary amount?

A more formal approach is the *algebraic method*, which involves writing two equal ratios and solving the equation. The equation can be written in several ways, but it is usually easier if you ensure that the unknown is a numerator (i.e. on the top line). To illustrate this, consider just the last example:

either  $\frac{k}{300} = \frac{8}{5}$ , leading to  $k = 300 \times \frac{8}{5}$

or  $\frac{k}{8} = \frac{300}{5}$ , leading to  $k = 8 \times \frac{300}{5}$ .

**To summarise**, numerical problems on ratio and proportion often reduce mathematically to an equation of the form  $\frac{a}{b} = \frac{c}{d}$ , where three numbers are known and the fourth has to be found. Problems are usually solved by the scaling or by the unitary method.

### Exercise E

1. Peanuts cost 70p for 100 grams. What is the cost of 550 grams of peanuts?
2. Raisins cost 75p for 100 grams. I pay £6 for a bag of raisins. How many grams of raisins do I get?
3. The distance from Paris to our campsite near Bordeaux is 640 kilometres. 5 miles is approximately 8 kilometres. What is the approximate distance in miles from Paris to our campsite?
4. I paid 44 000 Italian lire for a present. 3120 lire equal £1. What was the cost of the present in pounds and pence (to the nearest penny)?
5. A paint manufacturer makes green paint by mixing blue, yellow and white paint in the ratio 9 : 5 : 2.
  - (a) How many litres of yellow and white paint are needed to mix with 450 litres of blue paint?
  - (b) How many litres of each colour are there in 2000 litres of green paint?

# Answers to exercises

## Exercise A

1.  $\frac{1}{4}$
2.  $\frac{1}{6}$  (1 divided into 6 parts rather than 7 parts)
3.  $\frac{9}{7}$  or  $1\frac{2}{7}$  (considering only sevenths)
4.  $1\frac{2}{7}$  or  $1\frac{5}{7}$
5.  $\frac{9}{5} = 1\frac{4}{5}$
6. For example:

1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{5}$ (i.e. $3 \div 5$ or $\frac{1}{5}$ of 3) = 3 lots of $\frac{1}{5}$
2	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	
3	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	

7.  $\frac{2}{3}$
8.  $\frac{1}{4}$

## Exercise B

1. For example, 'The wingspan of the Angel of the North is roughly  $\frac{1}{6}$  of the width of the Millennium Dome', or 'The ratio of the width of the Millennium Dome to the wingspan of the Angel of the North is roughly 6 : 1'
2. (a)  $\frac{5}{8}$  (b)  $\frac{8}{5}$
3.  $\frac{3}{20}$
4.  $36 \div 4.5 = 8$  people
5.  $\frac{3}{8} \times \text{£}160 = \text{£}60$  and  $\frac{5}{8} \times \text{£}160 = \text{£}100$
6.  $q$  is  $\frac{5}{3}$  of  $p$
7.  $\frac{3}{5} \times 45 = 27$  litres
8. (a)  $\frac{5}{4}$   
 $\frac{5}{4} \times \text{£}20\,000 = \text{£}25\,000$   
(b)  $\frac{4}{5}$   
 $\frac{4}{5} \times \text{£}30\,000 = \text{£}24\,000$

## Exercise C

1.  $\frac{6}{16}, \frac{9}{24}, \frac{12}{32}, \dots$
2.  $\frac{4}{3}, \frac{12}{9}, \frac{16}{12}, \dots$
3.  $\frac{8}{9}$
4.  $\frac{5}{6} > \frac{4}{5}$
5.  $\frac{4}{7} > \frac{5}{9}$
6.  $\frac{71}{63}$  or  $1\frac{8}{63}$
7.  $\frac{1}{63}$

## Exercise D

- (a) No, because  $\frac{12}{3} = 4$  and  $\frac{20}{4} = 5$

(b) Yes, because  $\frac{6}{48} = \frac{21}{168} = \frac{1}{8}$

(c) No, because  $\frac{5}{15} = \frac{1}{3}$  and  $\frac{10}{20} = \frac{1}{2}$

(d) Yes, because  $\frac{28}{7} = \frac{96}{24} = \frac{16}{4} = 4$
- (a) No, because  $\frac{450}{150} = 3$  and  $\frac{350}{100} = 3.5$

(b) Yes, because  $\frac{375}{150} = \frac{250}{100} = 2.5$

Scale factor of enlargement is 2.5
- (a)  $m = 4$  ( $y$  changes by 4 for every 1 of  $x$ ). Not a proportion, because  $\frac{6}{1} = 6$ ,  $\frac{10}{2} = 5$ , etc.

(b)  $m = 0.2$ . A proportion, because  $\frac{0.2}{1} = \frac{0.4}{2} = \dots = 0.2$

(c)  $m = -2$ . Not a proportion, because  $\frac{12}{1} = 12$ ,  $\frac{10}{2} = 5$ , etc.

(d)  $m = \frac{8}{5}$  ( $y$  changes by 8 for every 5 of  $x$ , or by  $\frac{8}{5}$  for every 1 of  $x$ )

A proportion, because  $\frac{8}{5} = \frac{16}{10} = \dots = 1.6$

Equations are (a)  $y = 4x + 2$  (b)  $y = 0.2x$  (c)  $y = 14 - 2x$  (d)  $y = \frac{8x}{5}$
- (a) Yes – cost of tomatoes is proportional to quantity purchased.

(b) Yes – distance travelled is proportional to the time on the journey.

(c) No – because of the fixed charge of £5 (but it is a linear relationship).

(d) Yes – distance in kilometres is proportional to distance in miles.

## Exercise E

- Cost =  $\frac{550}{100} \times 70 = 5.5 \times 70 = 385\text{p}$

(By scaling, or  $\frac{70}{100} \times 550$  by unitary method.)
- Quantity =  $\frac{100}{75} \times 600 = 800\text{ g}$

(By unitary method, or  $\frac{600}{75} \times 100$  by scaling.)
- 8 km = 5 miles (approx), so 1 km =  $\frac{5}{8}$  miles, 640 km =  $\frac{5}{8} \times 640$  miles = 400 miles (unitary method)
- Cost (£) =  $44\,000 \div 3120 = \text{£}14.10$

(Scaling, which is the only sensible method here.)
- (a) Yellow paint:  $\frac{5}{9} \times 450$  litres = 250 litres

White paint:  $\frac{2}{9} \times 450$  litres = 100 litres

(Start from 9 parts being 450 litres and use unitary method.)

(b) Blue paint:  $\frac{9}{16} \times 2000$  litres = 1125 litres

Yellow paint:  $\frac{5}{16} \times 2000$  litres = 625 litres

White paint:  $\frac{2}{16} \times 2000$  litres = 250 litres

(9 + 5 + 2 = 16 parts. Start from 16 parts being 2000 litres.)

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