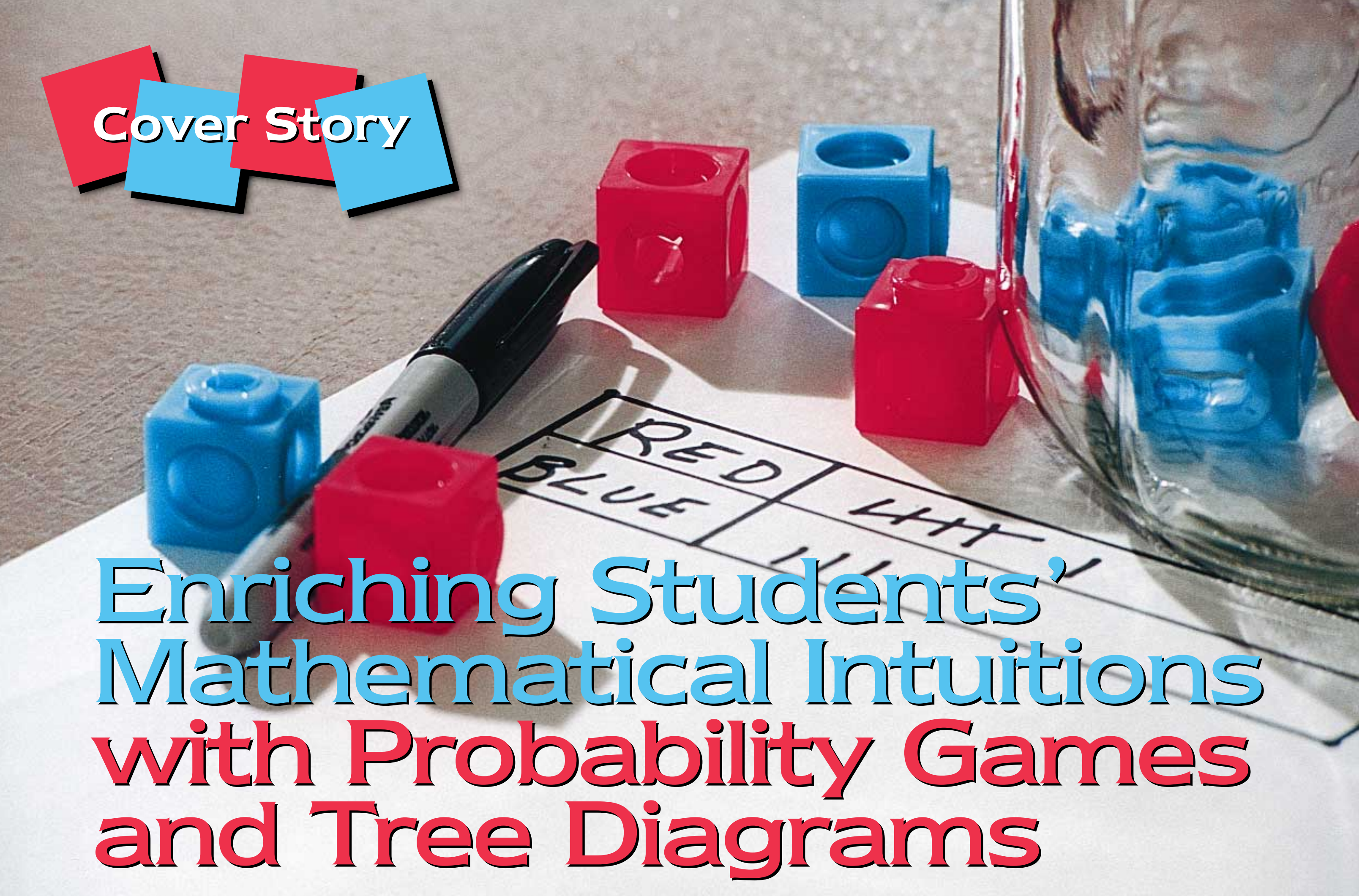


Enriching Students' Mathematical Intuitions with Probability Games and Tree Diagrams

LESLIE ASPINWALL AND KENNETH L. SHAW

> Intuitive thinking, the training of hunches, is a much-neglected and essential feature of productive thinking not only in formal academic disciplines but also in everyday life.
>
> —*Jerome S. Bruner* (1960, 13–14)

MIDDLE SCHOOL STUDENTS OFTEN develop games to help make such decisions as who goes first or who gets the biggest piece of cake. Many students seem to have good intuition for determining which games are fair. Students feel comfortable with flipping a coin or playing "odd it out" to determine a winner. One easy visual way to determine the fairness of flipping a coin is by using a tree diagram (see **fig. 1**). The diagram shows that this game offers an equal probability for either player to win, assuming that the outcomes are equally likely.

When playing "odd it out," players count aloud, "one, two, three," shaking their fists down on each count. On the count of "three," each player holds out either one or two fingers, producing two results: either the total number of fingers is even or the total number is odd. This game is somewhat more complicated than flipping a coin, but most middle school students intuitively understand that "odd it out" produces the same chance of attaining even and odd outcomes and, thus, believe that the game is fair. This intuition can be verified by looking at the tree diagram in **figure 2,** which shows two

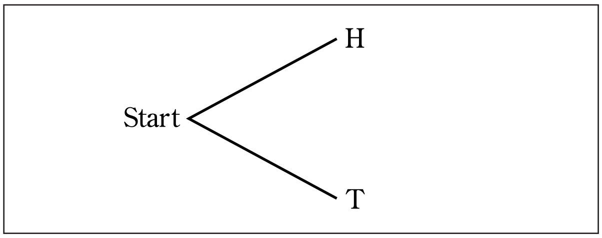


Fig. 1 Tree diagram to show fairness of a coin toss

LES ASPINWALL, *aspinwal@coe.fsu.edu, teaches at Florida State University, Tallahassee, FL 32306. His research areas include mathematical writing, visualization, and probability and statistics.* KEN SHAW, *kshaw@mail.pc.fsu.edu, teaches mathematics education courses and was serving as interim associate dean at Florida State University, Panama City Campus, Panama City, FL 32405, at the time this article was written. His research interests involve investigating the impact of teachers' beliefs on teacher learning and change.*

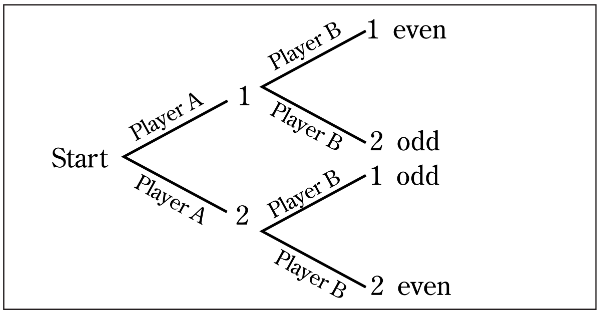


Fig. 2 Tree diagram to show fairness of "odd it out"

even and two odd outcomes. Because all the outcomes are equally likely, the probability of obtaining an even or an odd outcome is the same; therefore, the game is fair.

Bruner (1960) suggests several strategies for fostering intuition. Two of these strategies involve students playing games that require them to make inferences from partial information. The games force students to go beyond the given data, then check for consistency. This article describes how these two strategies were used to determine and nurture four eighth-grade students' mathematical intuitions as they relate to fairness. The students, Ahmed, Bill, Clara, and Denise (all pseudonyms), attend a large middle school consisting of seventh and eighth grades in middle Tennessee.

We started the interviews by asking the students about flipping a coin and playing "odd it out." All four students believed that players had an equal chance of winning either game. We set out to explore how these students' intuitions about fairness might influence their thinking as the games became progressively more difficult.

"Red versus Blue"

THE "RED VERSUS BLUE" GAME CONSISTS OF four activities (see **fig. 3**). We started the "red versus blue" game with a warm-up question.

Interviewer. If you flipped a coin one hundred times, how many times would you expect to get a result of heads?

Bill and Clara. Fifty times. [Ahmed and Denise nod in agreement.]

This answer suggests that the students have formed intuitive schemes, gleaned from experience, for deciding whether a game is fair.

After we introduced the "red versus blue" game, we asked the students to play the game in pairs. The rules for each of the first two activities were explained. The teams should take turns drawing a cube from the bag and examining and recording its color. Each team should draw and replace the cube twenty-five times.

The students' combined result for activity 1 was twenty-six reds and twenty-four blues, and their combined result for activity 2 was twenty-five reds and twenty-five blues. We began the discussion by asking which color the students thought would win in each game. For activity 1, Denise and Bill spoke for the group: "It's equal, one blue and one red. It would be different if there were more of one color." When asked whether red or blue would win in activity 2, Denise stated, "Still can't tell." The others all nodded in agreement. Subsequent discussions revealed that Bill and Denise characterized a game as fair if they could not predict a winner. By saying,

Red and blue cubes are placed in a bag, and two players or teams take turns reaching into the bag without looking, removing a cube, then replacing it. Work in pairs or in groups of four with two players on each team.

Materials

- Bag or vase
- Red and blue cubes

Game rules

- One team is the red team, and the other is the blue team.
- Red and blue cubes are placed in a bag.
- Teams take turns reaching in without looking, removing one cube, examining and recording its color, and replacing it.
- Teams record their scores by placing a tally mark on tally sheets, such as the following:

Red	||||
Blue	||

- Ask students to play a predetermined number of rounds as a test to determine whether the game is fair.
- The winner is the team that has drawn the most cubes matching the team color.

Activities for "red versus blue"

1. Place one blue cube and one red cube in the bag.
2. Place two blue cubes and two red cubes in the bag.
3. Place two blue cubes and one red cube in the bag.
4. Place three blue cubes and one red cube in the bag.

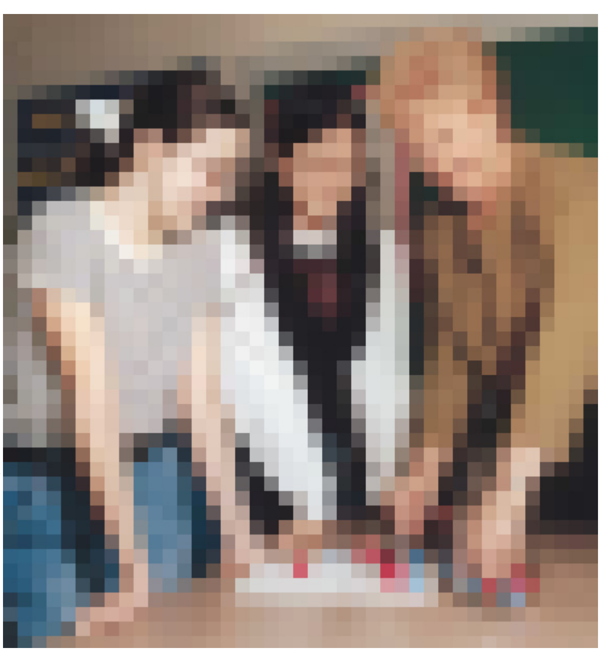
Fig. 3 Instructions for "red versus blue" game

"Still can't tell," Denise meant that anyone can win and the game is fair. Before playing these new games, all four students expressed an expectation that a game would be fair if the outcomes were "equal," that is, if the games offered the same number of equally likely outcomes for each result. None of the students experienced difficulty in deciding that these games were fair; moreover, they all determined that their results verified their initial expectations.

Activity 3, using two blue cubes and one red, and activity 4, using three blue cubes and one red, added a new twist to the games because both are unfair. Before playing activity 3, Bill and Clara intuitively felt sure that the blue team would win and that the game was not fair. Although Denise agreed, she was less enthusiastic in her response, and Ahmed was unsure. Bill, Clara, and Denise had developed a scheme for deciding that the outcomes for blue exceeded those for red, so the two outcomes were no longer equally likely. Denise had begun to quantify her intuition as she extended a previous statement, "There are more blue [cubes than red cubes]."

Students' intuitions became more sophisticated

Students' combined result for activity 3 was fifteen reds and thirty-five blues, and the combined result for activity 4 was fourteen reds and thirty-six blues. After the games, students' remarks suggested that all four students had clarified their thinking. Bill and Clara, who intuitively thought that the blue team would win in activity 3, thought the same for activity 4. Ahmed and Denise, who were unsure about activity 3, were more confident about activity 4. When we asked what features of a cube-drawing game guarantee that the game is fair, all four students said, "The number of cubes should remain the same." For the unfair games, the students responded, "There were more blue cubes [than red cubes]." Ahmed, Bill, Clara, and Denise had developed insight, and their intuition had become more sophisticated as they began to add a numerical dimension to their answers for activities 3 and 4. Consider the discussion that occurred immediately before activity 4:



Interviewer. For activity 4, we have three blue cubes and one red cube. Ahmed [who had been relatively quiet and slower to respond], what do you think the outcome of this game will be?

Ahmed. I think blue will win. Yeah, there are more blue cubes.

Bill. About three times as many blues as reds.

Interviewer. So, out of our fifty total, what . . . ?

Bill. [Figuring] About seventeen reds. . . .

Clara. And about thirty-four blues.

Tree diagrams for the four activities clearly illustrate which games are fair. Activity 1 **(fig. 4)** allows only one outcome for red and one outcome for blue; activity 2 **(fig. 5)** allows two outcomes for blue (B_1 and B_2) and two outcomes for red (R_1 and R_2). Nonetheless, the number of outcomes for each color is equal and each outcome is equally likely.

Activity 3 **(fig. 6)** has two outcomes for blue (B_1 and B_2) but only one outcome for red, and all three outcomes are equally likely, making the game unfair. Activity 4 **(fig. 7)** is also unfair because it allows three outcomes for blue (B_1, B_2, and B_3) and only one outcome for red. Ahmed, Bill, Clara, and Denise all agreed that the tree diagrams helped them confirm the fairness of the games in these activities.

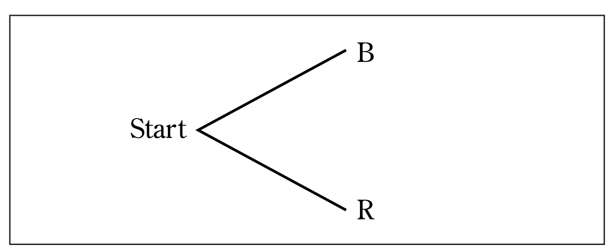


Fig. 4 Tree diagram for activity 1

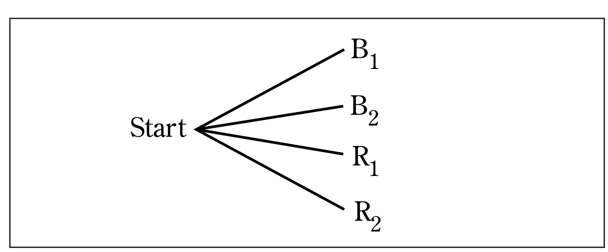


Fig. 5 Tree diagram for activity 2

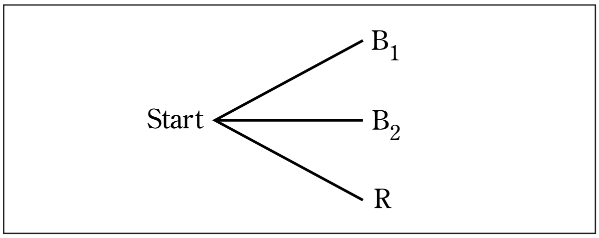


Fig. 6 Tree diagram for activity 3

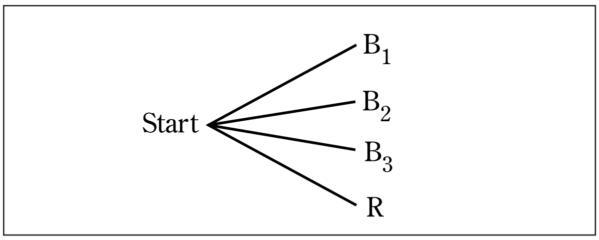


Fig. 7 Tree diagram for activity 4

"One Color versus Two Color"

THESE STUDENTS HAD LITTLE DIFFICULTY IN DEtermining whether a game is fair when it involved selecting one colored cube. We decided to ask students to select two cubes. When using the tree diagrams, this task is equivalent to selecting one cube, then selecting another without replacing the first cube. For activities 5, 6, and 7 (see **fig. 8**), each team is designated as one color or two color. If the two cubes drawn are the same color, the one-color team wins the round. If the two cubes drawn are different colors, the two-color team wins.

We explained the new game and asked the students to predict the outcome for activity 5, which uses two blue cubes and one red cube. The following discussion took place as the students decided that the game was unfair:

Bill and Denise. There should be more two color.
Clara. No, there should be more one color.
Bill. There's two ways to get one red and one blue, but there's only one way to get two blue.
Clara. Oh, yeah!
Interviewer. Do you have agreement?
All. There should be more two color.

The students performed twenty-five trials each, and the combined result for activity 5 was twenty one-color draws and thirty two-color draws. We asked the students whether this result was what they had expected, and Bill said, "Yes." Denise responded, "Yeah, I think so." Ahmed and Clara nodded in agreement. The students believed that they were correct, but their earlier intuition was becoming less reliable. They disagreed about the possible outcomes before reaching a consensus, because they were developing ideas about why these outcomes might occur as they did. Before performing the experiments for activity 6, we ask the students some preliminary questions:

Materials
- Bag or vase
- Red and blue cubes

Game rules
- One team is the one-color team, and the other is the two-color team.
- Red and blue cubes are placed in a bag.
- Teams take turns reaching in, again, without looking; removing two cubes; examining and recording whether the cubes are one color or two colors; and replacing the cubes.
- If the colors are the same—two reds or two blues—the one-color team scores a point. If the colors are different—one of each color—the two-color team wins. Again, teams record their scores by placing a tally mark on tally sheets, such as the following:

One color	||||
Two color	||

- Ask students to play a predetermined number of rounds as a test to determine whether the game is fair.
- The winner is the team with the most tally marks.

Activities for "one color versus two color"
5. Place two blue cubes and one red cube in the bag.
6. Place two blue cubes and two red cubes in the bag.
7. Place three blue cubes and one red cube in the bag.

Note: In playing the "one color versus two color" game, children's, and even adults', intuition can sometimes cause difficulty. This difficulty may be the result of previous experience with such games as tossing a coin for heads or tails; playing "odd it out," described previously; or completing activities similar to those in the "red versus blue" game, also described previously. Sometimes, having students check their intuition using tree diagrams may help clarify the fairness of the activities.

Fig. 8 Instructions for the "one color versus two color" game

Interviewer. If I put two blues and two reds in the bags, would you expect this game to be fair if we are examining one color versus two color?

Bill. Yes; two ways to make each.

Interviewer. What do you mean?

Bill. For one color, you can get two blues or two reds—two ways. For two color, you can get a red and a blue or a blue and a red. Still two ways.

The game is unfair, however, and the students' combined result for activity 6 was fifteen one-color draws and thirty-five two-color draws. When we asked what they thought about this result, Bill said, "I don't think that's right." Ahmed, Clara, and Denise agreed with Bill, stating, "But the colors were matched [two blue cubes and two red cubes]."

We also began activity 7 with a preliminary question:

Interviewer. What if I put three blue and one red; would that be a fair game for two color versus one color?

All. No! [Thinking out loud] One color . . . , no, two color, ...

Bill. Wait, one color would win.

Clara and Denise. Yeah.

Students' experience influenced their thinking

Bill, Denise, and Clara thought that the one-color team should win because more blue cubes were included than red. We left them alone for a few minutes to think, and they began talking to one another about matching each blue with each of the other two blues and the red. The students were beginning to improve their intuition as they attempted to analyze the possible outcomes.

Clara. I think blue just because there's more of them.

Interviewer. Blue is not one of the choices; it's one color or two color.

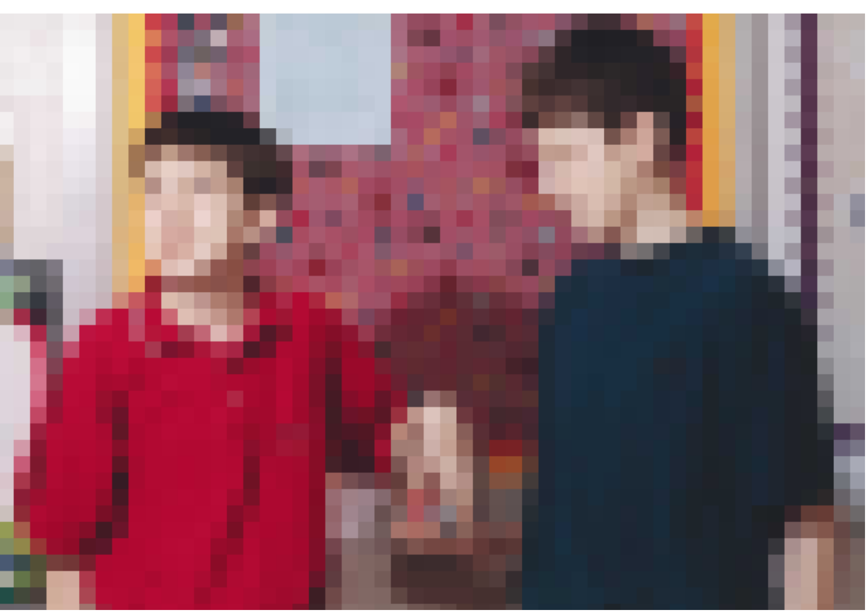
Clara. Uh, oh yeah. I mean one color.

Bill and Denise. Yeah, more one color because there are so many blues and only one red.

The students' experience with flipping a coin, playing "odd it out," and doing activities 1–4 seemed to influence their thinking. Clara revealed this influence by saying "blue" instead of "one color." Their initial intuition would prove to be misleading in this game, which is fair. After the students performed twenty-five trials each, the combined result for activity 7 was twenty-four one-color draws and twenty-six two-color draws. When we asked whether they thought the game was fair, Bill said, "I don't think it is. I just think it didn't come out right. It's just luck. In any game, anybody could win." Ahmed, Clara, and Denise responded, "I don't know [whether the game is fair]."

We then showed the students the tree diagrams for activities 5 and 6 (see **figs. 9** and **10**).

Bill. I was right with two blues and one red [activity 5], but it still seems impossible for two blues



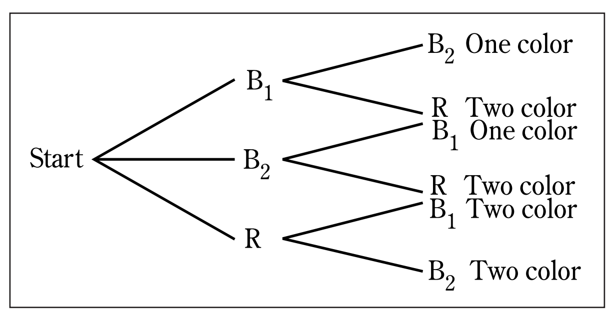


Fig. 9 Tree diagram for activity 5

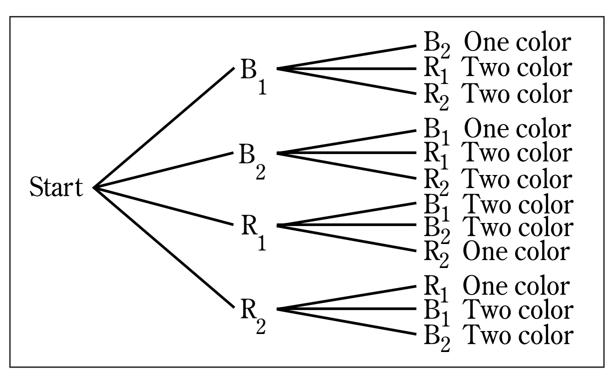


Fig. 10 Tree diagram for activity 6

and two reds [activity 6]. That should be equal.

Interviewer. Yes. Let's look at [the tree diagram for] three blues versus one red [activity 7].

Clara. I still don't know.

Bill. [After thinking about the tree diagram], I think now it should be equal.

Interviewer. What do you think, Denise?

Denise. I'm having trouble visualizing it.

The students began debating among themselves. Their conversation was directed toward the concept of tree diagrams, not intuition. Initially, they thought that activity 7 could be fair because of the larger number of blue cubes. Now they were wavering because they had a new way of analyzing the activity using tree diagrams. In other words, they were no longer relying on a quick, intuitive reaction alone but were allowing their intuition to be influenced by more thorough analysis. The group had the following discussion after analyzing the tree diagram for activity 7 **(fig. 11).**

Bill. Yeah, now I'm right [that the game is fair], but it doesn't seem like it should be that way. Most people would think that one color should win.

Ahmed, Clara, and Denise. Yeah.

Interviewer. What does it take to determine when a game like this is fair?

Bill. Brainwork. You have to count the number of ways it can happen [when the outcomes are equally likely]. This would be a good con.

Activity 7 asked students to work with three blue cubes and one red cube in the bag. With a bit less confidence than they had exhibited in previous activities, the students said that the one-color group would win. Almost uniformly, they reasoned that because the activity included more blue cubes than red cubes, the results would likely favor the one-color team. After they played the game and analyzed the tree diagram, they were left scratching their heads.

After the whole-group discussion, students were

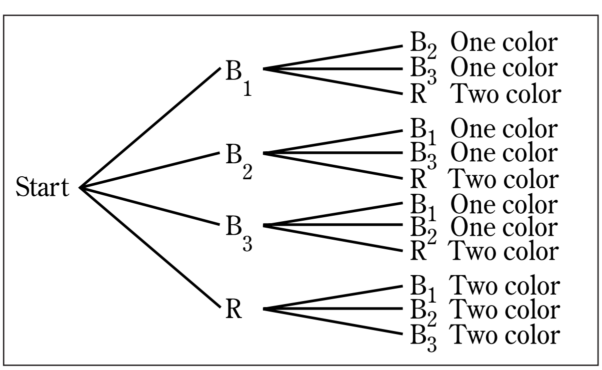


Fig. 11 Tree diagram for activity 7

asked what the difference was between selecting two cubes and selecting one cube. What features would guarantee that the game would always be fair when selecting two cubes? This last question was difficult for the students to answer. The tree diagrams for activities 5, 6, and 7 proved to be enlightening visual tools for the students, helping them to modify their thinking and intuitions about these types of activities.

Summary

WHEN WE ASKED THE GROUP ABOUT THE FAIRness of familiar games, they demonstrated their intuition about the outcomes. They insisted, for example, that flipping a coin is fair. Bill even hinted at probabilistic reasoning when he said, "There's a fifty-fifty chance of heads or tails." When asked about another game with which they were familiar, "odd it out," none was able to explain in detail why this game is fair, other than to say that the outcome "could be odd or even."

Before playing the game in activity 6, students' intuitions made them confident that the game would be fair. After the activity, the students were shown the tree diagrams and were involved in a discussion about their findings; they thought that their results were anomalous. Clearly, their mathematical intuitions were being stretched, and, in fact, they tended to believe their intuitions despite multiple samples and the graphic example of the tree diagram.

Similar reactions were also evident in activity 7. When their results with the "one color versus two color" game, using three blue cubes and one red, suggested that the game was fair, Bill said, "It did not come out like I said it should. I don't think it is [fair]. I just think it didn't come out right." He then suggested his understanding of the sampling error of an experiment with a small sample size when he said, "It's just luck. In any game, anybody could win."

NCTM's *Principles and Standards for School Mathematics* (2000) has already begun the call for reform in middle school mathematics instruction with a standard devoted to data analysis and probability. NCTM contends that students should "understand and apply basic concepts of probability" to enable all students to compute probabilities for simple compound events, using such methods as organized lists, tree diagrams, and area models" (p. 248).

As we have seen, students need a good deal of experience with such tasks to assist them in overcoming primitive intuitions. As teachers, we need to understand that students' intuitions are powerful constructs and may adversely affect teaching and

Cover Story

learning, regardless of the clarity and logic of instruction. The tree diagrams assisted the students visually in understanding the outcomes and probabilities.

We have used these games both as a focus for small-group activities and as a protocol for interviews. As cooperative group activities, either the experiments can be divided up and performed by different groups or each group can perform the same experiment. For each activity, students play a predetermined number of rounds and discuss the results. They determine whether one outcome will usually win or whether the game is fair. After the experiments, students share their results with the class on a master table on the chalkboard. A whole-group discussion takes place to determine whether each activity is fair. At this point, constructing tree diagrams provides a powerful visual model for students to use in determining the fairness of a game. To help students focus on their intuitions, the teacher may wish to ask such questions as these:

- Which games are fair?
- Why do you think the games are fair or not fair?
- How have your ideas changed after seeing the results of all the games?
- How do tree diagrams help you decide whether a game is fair?
- How do you make the unfair games fair?

The teacher may wish to instruct the students in three stages during the activities:

1. Students begin by using only their intuitions to decide whether a game is fair.
2. Next, students perform an experiment, consisting of a number of trials, and use the results to confirm or modify their intuitions.
3. Students construct their own tree diagrams, including all outcomes for the game, to enrich their intuitions.

The activities, in the order presented, could easily be used in a whole-class setting or as a theme in a learning center. The first four activities build on one another and can evoke productive discussions about such probability concepts as outcomes, data, chance, and fairness. The last three activities may be somewhat foreign to students because the results may be counterintuitive. However, these activities may cause the conflict necessary for students to reflect intently on their intuitions. This reflection, along with the tree diagrams and discussions among students and teacher, may prompt students to begin to redefine their intuitions and, thus, gain a better understanding of probability.

Bibliography

Bruner, Jerome S. *The Process of Education.* Cambridge: Harvard University Press, 1960.

Burns, Marilyn. *Is It Fair? A Look at Probability.* White Plains, N.Y.: Cuisenaire Co. of America, 1995.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics:* Reston, Va.: NCTM, 2000.

The authors wish to thank teacher Pat Wilson and principal William H. (Butch) Vaughn at Central Middle School in Murfreesboro, Tennessee, for their assistance with this article.