Providing Opportunities to Learn Probability Concepts

James E. Tarr

NCTM's *Principles and Standards for School Mathematics* (2000) identifies Data Analysis and Probability as one of the five content standards for pre-K–12 mathematics and delineates learning expectations at each of four grade bands. This standard places much more emphasis on data analysis than on probability, particularly for grades pre-K through 5. Indeed, only one of the four goals in the standard directly addresses probability, and no probability learning expectations are explicitly stated for grades pre-K through 2. The standard states, however, that "instructional programs from prekindergarten through grade 12 should enable all students to understand and apply basic concepts of probability" (p. 48).

The Big Ideas

Is the message of giving all students the opportunity to learn probability concepts compatible with the limited emphasis that *Principles and Standards* places on probability in prekindergarten through grade 5? This article highlights some of the "big ideas" for probability in elementary school through discussions and activities.

Probability is a measurement of the likelihood of future events

Young children enter school with intuitive notions of likelihood, as evidenced by such remarks as "She always gets her way!" "I usually guess wrong," and "That will never happen." Such comments provide a foundation on which to develop an understanding of the relative likelihood of future events. More specifically, primary school children should learn to distinguish events that are certain from those that are not and to estimate the relative likelihood of events that are neither certain nor impossible.

Teachers can begin a discussion by asking, "Can you think of events that are certain to happen?" Some children may respond, "I am certain that I had cereal for breakfast this morning" or "I am certain that I was born." However, since the power of probability is that it allows us to estimate the likelihood of events to happen in the future, it is necessary to distinguish between what has already happened and what may occur in the future. At the opposite end of the likelihood spectrum is the concept of an impossible event, and students may offer vivid descriptions of future events that cannot happen. After establishing the parameters of likelihood, groups of students can generate lists of events that are neither impossible nor certain. They

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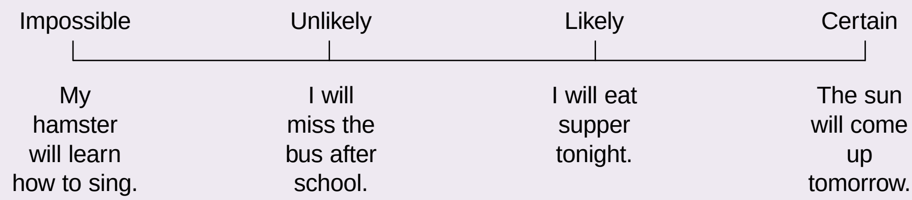
Edited by **Jeane Joyner,** *jjoyner@dpi.state.nc.us, Department of Public Instruction, Raleigh, NC 27601, and* **Barbara Reys,** *reysb@missouri.edu, University of Missouri, Columbia, MO 65211. This department is designed to give teachers information and ideas for using the NCTM's* Principles and Standards for School Mathematics *(2000). Readers are encouraged to share their experiences related to* Principles and Standards *with* Teaching Children Mathematics. *Please send manuscripts to "Principles and Standards,"* TCM, *1906 Association Drive, Reston, VA 20191-9988.*

FIGURE 1

Placing events on a likelihood line



can trade events with other groups and discuss where the events should be placed on a "likelihood line" (see **fig. 1**). Teachers can then suggest a list of ordinary events (e.g., "There will be a fire drill today") and ask students to determine where the events might go on the likelihood line. In this informal study of probability, the notions of *more likely* and *less likely* should come from students' experiences rather than mathematical situations, such as rolling dice or drawing names from a hat.

In upper elementary school, students should begin to associate the word *probability* with how likely an event is to occur and make the transition from verbal descriptions of likelihood to numeric descriptions of probability. In particular, they should learn how to quantify likelihood and understand that a single number can represent the measure of the likelihood of an event. Initially, part-part comparisons (e.g., *odds*) will allow students access to probability problems, but eventually, they should come to use part-whole comparisons to describe probability. Students should also understand that 0 is used to represent the probability of an impossible event and that 1 is used to represent the probability of a certain event. The likelihood of an event that is neither impossible nor certain can be represented as a number between 0 and 1 in the form of a fraction, decimal, or percent. Students should learn to assign conventional numerical probabilities to situations involving only a few possible outcomes and be able to convey the relative likelihood of events by placing probabilities on a number line from 0 to 1 (see **fig. 2**).

The concept of sample space

Sample space is the set of all possible outcomes and their associated probabilities. The ability to specify the exhaustive set of events that might happen in a given situation is fundamental to understanding probability. Primary school children exhibit mixed success in identifying the complete set of outcomes in a one-stage experiment because some are reluctant to acknowledge that an event can occur, particularly when the prevailing odds are against the occurrence of such an event. With respect to two-stage experiments, children as young as seven are able to formulate efficient strategies for listing ordered pairs and ordered triples when using manipulative materials (English 1993). Accordingly, elementary school students should explore systematic ways (e.g., lists, tables, or tree diagrams) to generate the complete sample space. Without such opportunities, some children may be inclined to deny the existence of particular sequences in a series of repeated trials of an experiment. Moreover, consideration of the total number of elements in the sample space is essential in making part-whole comparisons and assigning numerical probabilities (Piaget and Inhelder 1975).

Upper elementary school students need to develop an understanding of a related principle, namely that

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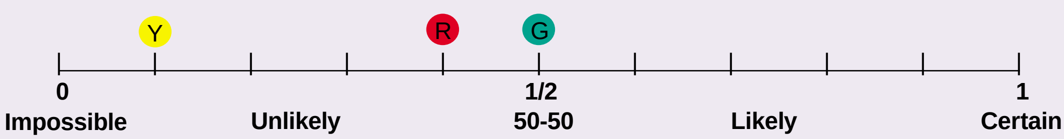
FIGURE 2

Representing likelihood as a number between 0 and 1

A gumball machine contains 1 yellow, 4 red, and 5 green gumballs. Taylor is hoping for a green gumball. Predict what kind of gumball Taylor will get by placing a G, R, and Y somewhere on the scale from 0 to 1 to indicate her chances of getting each of the three types of gumballs.



the sum of the probabilities of all sample space outcomes is 100 percent or 1. Although this fact may be obvious to teachers, students may exhibit misconceptions regarding sample space. For example, in making probability estimates, some may reason, "I have a 50 percent chance of getting red, a 40 percent chance of getting blue, and a 30 percent chance of getting yellow." With an upper bound of 100 percent, the likelihood line—and in particular, the task of placing probabilities of events on it—may help to promote the principle that the sum of the probabilities of all outcomes in the sample space is 1 (or 100%).

Predicting outcomes of simple experiments and testing the predictions

Probability is sometimes regarded as the least intuitive branch of mathematics because people of all ages have difficulties developing correct intuitions about fundamental ideas in probability (Garfield and Ahlgren 1988). Evidence suggests that students are better able to develop correct intuitions when they are introduced to probability using simulations of random phenomena (Shaughnessy 1992). Students in the upper elementary grades should predict how often an event will happen in a given number of trials and should test their predictions. For example, they can predict how many times a coin will land on heads when tossed eight times. Most students expect four heads in a sample of eight tosses, and many indicate that only two heads would be an unusual result. After carrying out the simulation, students should compare what was expected with what really happened. In doing so, they will come to realize that four heads is the best estimate in the sense that it is the most likely outcome; more often, a number other than four is obtained. Moreover, they may come to learn that outcomes they perceived to be unusual (e.g., two heads) can, in fact, occur; or stated alternatively, events associated with low probabilities can and do occur. Finally, carrying out the simulation again is an effective means of promoting another important principle, namely, that repeated simulations yield a variety of results, thus reinforcing concepts of randomness and variability.

Although probability is a subject in its own right, it is closely related to other branches of mathematics

Forming probability estimates using empirical data

The idea of randomness is that the results of a single trial are not predictable but that a predictable pattern can be found to the results of many trials. Thus, through data collection we can estimate probabilities, particularly when we carry out a large number of trials. In contrast to rolling dice and flipping coins, some situations are possible for which the probability of an event can be determined only through data collection. For example, the probability of a randomly selected person having brown eyes is determined only by collecting data from the general population. Likewise, the probability that a toothpaste cap will land on its side when dropped cannot be determined theoretically, since it would be impossible to characterize the mechanics involved as the cap tumbles. In these situations, data must be collected so that inferences regarding likelihood can be made. In general, larger samples are more likely to produce better estimates of the probability that such events will occur.

Students can investigate whether a button is equally likely to land on each of its two sides. Some may believe that because the sample space contains two elements, each event must have a probability of 1/2. Many will likely put too much faith in small samples and, after tossing a button only a dozen times, erroneously conclude that it is prone to land on one particular side more than the other because "twelve times proves it!" Although no fixed number of trials can determine the exact probability of the button's landing on each of its sides, the likelihood that the probability estimated from a sample is close to the actual probability increases as the number of trials increases. The need to conduct a large number of trials is one of the reasons that probability is not emphasized in the early grades in *Principles and Standards*. Children need to be familiar and at ease with large numbers. Using a small number of experiments to estimate probabilities can be very misleading. Children also need to be familiar with fractions to deal with relative frequencies. In the upper elementary grades, such tasks as tossing a button provide opportunities to engage students in the process of statistical inference as they develop the notion of the probability of an event and formulate probability estimates.

Connecting Probability with Other Strands of the Curriculum

Although probability is a subject in its own right, it is closely related to other branches of mathematics. Accordingly, opportunities to learn probability concepts can arise from the study of number, geometry, and data analysis. Moreover, *Principles and Standards* advocates that probability activities

FIGURE 3

Comparing the magnitude of 4/9 and 3/8 using the context of probability

Selecting the first name	Selecting the second name
Bill, Sergio, Marshall, Alfinio	Bill, Marshall, Alfinio
Ericka, Amanda, Rose, Pam, Sheri	Ericka, Amanda, Rose, Pam, Sheri
Selecting the first name: P(boy) = 4/9	Selecting the second name: P(boy) = 3/8

should reinforce conceptions in other content standards, as illustrated in the following examples.

Probability and rational numbers

The Number and Operation Standard states, "Through the study of various meanings and models of fractions . . . students gain facility in comparing fractions" (NCTM 2000, 149). Although number lines and area models are commonly used in comparing fractions, probability, too, can be an effective means for comparing magnitude. Consider the fractions 4/9 and 3/8. Finding a common denominator (72, in this instance) or representing each fraction as a decimal are typical strategies for comparing the pair of rational numbers. Next consider the numbers in the context of probability: Two students will be selected at random to become group leaders. The names of nine students—four boys (red) and five girls (blue)—are put in a hat. In the first drawing, a boy's name is selected; now the hat contains the names of three boys and five girls. Has the probability of selecting a boy's name changed? (See **fig. 3.**) Before gaining facility with rational numbers, many students use part-part reasoning to make probability comparisons. In particular, they may argue that in the first drawing, four boys' names and five girls' names are in the hat, but after a boy's name is selected, only three boys' names and five girls' names are in the hat. In essence, the number of girls' names remains constant while the number of boys' names decreases. The probability of selecting a boy's name in the second drawing has decreased from 4/9 to 3/8. Similarly, with fewer boys' names to compete with, the probability of selecting a girl's name has increased, thus making 5/8 larger than 5/9.

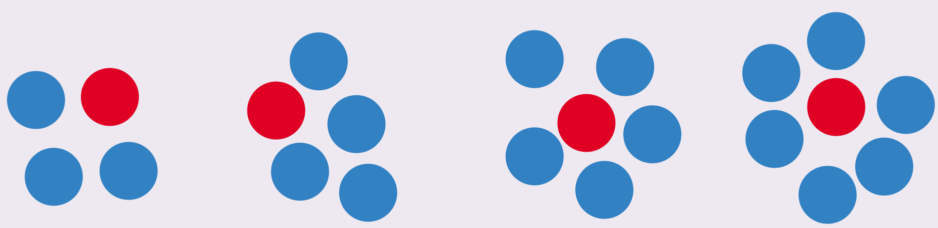
Likewise, density of the rational number system is an important mathematical principle. Children in upper elementary grades should "begin to understand that between any pair of fractions, there is always another fraction" (NCTM 2000, 150). This principle can be developed through connections with probability. The probability context enables students to identify fractions that approach 0 or 1. Consider a raffle in which only four tickets have been sold. The probability of winning the drawing is 1/4, and that of not winning the drawing is 3/4. As more tickets are sold, however, the probability of winning (selecting the red ticket) decreases, converging toward 0; while the probability of not winning (selecting a ticket other than red) increases, converging toward 1 (see **fig. 4**). This model may offer further help in ordering fractions.

Probability and data analysis

Ideas from probability serve as a foundation for data

FIGURE 4

Identifying fractions between 0 and 1/4 and between 3/4 and 1



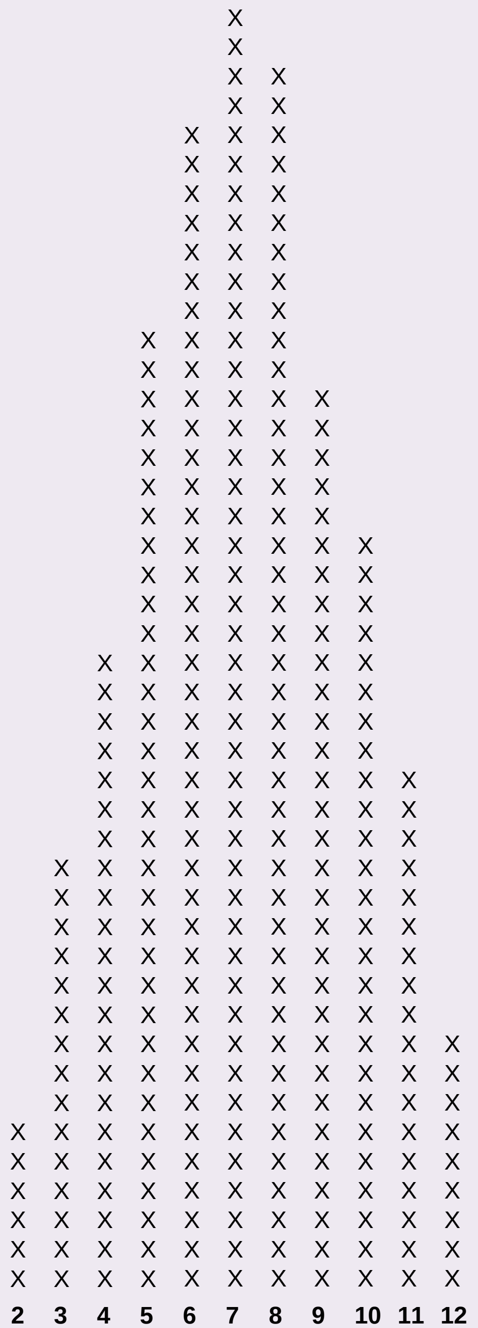
P(red):	1/4	>	1/5	>	1/6	>	1/7
P(blue):	3/4	<	4/5	<	5/6	<	6/7

The chance that a red ticket will be drawn *decreases* from left to right, while the chance that a blue ticket will be drawn *increases* from left to right.

analysis. The big ideas of central tendency, variability, and distribution can arise naturally from data collected from simulations of random phenomena; in collecting such data, students can explore probability and statistics simultaneously while fostering connections between data and chance. Some students, for example, may hold a pervasive belief that they are “good at rolling big numbers” because they may have recently won a game by doing so. To challenge this belief, students can collect data on repeated rolls of a die and learn that for large samples, the distribution of relative frequencies begins to appear uniform. By way of contrast, students can repeatedly roll two dice, determine the sum, and display the resulting data visually. If the data from individual simulations are combined, the distribution of outcomes in the large sample will begin to resemble a symmetrical staircase (see **fig. 5**). By focusing on the shape of the distribution, students find that the trends in the data become more apparent, as does the idea that not all outcomes are equally likely. Moreover, the activity affords the opportunity to convey another fundamental principle in data analysis, namely, that for a given event, the experimental probability (through repeated trials) is more likely to approximate the theoretical (actual) probability as the number of trials increases.

FIGURE 5

Distribution of pooled data from students’ simulation of rolling two dice



Probability and area

The opportunity to learn probability can readily arise from the study of geometry. After learning how to determine the area of any rectangular region, students can move on to solving more complex area problems, such as the one in **figure 6.** Consider an area problem set in the context of probability: Suppose that you and your friends use sidewalk chalk to draw the target in **figure 6.** You take turns blindly tossing a beanbag onto the target. One player scores a point if the beanbag lands on a white area and the other player scores if the beanbag lands on a shaded area. Is the game fair? Given that the beanbag lands at some random location on the target, do you predict that it will land in the white region or the black region? Teachers can promote quantitative reasoning by asking, “Can you use numbers to describe the likelihood of the beanbag’s landing in each region?” Some students are likely to make whole-number comparisons after determining the areas for the two possible outcomes. Others may use percents to compare the likelihood of each outcome. The problem represents a natural extension of two-dimensional geometry and effectively connects probability with geometry.

Probability and coordinate geometry

The game of Battleship may be used as a context for studying coordinate geometry. In this classic game, each player has five ships, each of which occupies a certain number of squares on the playing field: the Destroyer (two squares); the Submarine (three squares), the Cruiser (three squares), the Battleship (four squares), and the Aircraft Carrier (five squares). After placing their ships on the coordinate grid (see **fig. 7**), players take aim by calling ordered pairs. By playing the game, stu-

FIGURE 6

Where do you predict that the beanbag will fall?

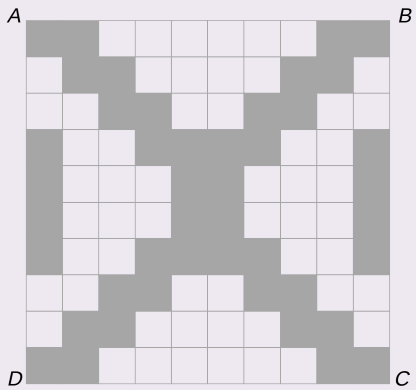
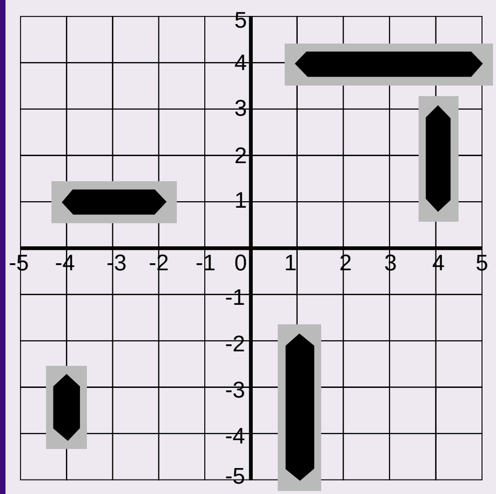


FIGURE 7

Using Battleship to study coordinate geometry and probability simultaneously



dents become efficient in naming points on the Cartesian coordinate plane. The following set of questions can extend the game:

- Do you predict that your first shot will score a hit on one of your opponent's ships? Why or why not? Can you use numbers to describe the likelihood of hitting one of the ships?
- Suppose that your first shot fails to hit one of the ships; has the likelihood of hitting a ship changed, or is the chance the same as it was before? Explain.
- Next suppose that you hit one of your opponent's ships. Will you choose your next ordered pair at random? Why or why not?

Questions such as these promote mathematical connections as students learn how geometry can be used to study the concepts of randomness, the probability of an event, the complement of an event, and even conditional probabilities.

Conclusion

Principles and Standards recommends that all students be given opportunities to learn probability concepts. Despite its designation as part of the core curriculum for school mathematics, however, probability is regarded as less important than other content strands, such as number and geometry, in the elementary grades. Probability topics generally appear in the latter portions of textbooks, and standardized tests typically allocate only a few items to probability. Ultimately, classroom teachers decide what topics are taught in mathematics. Charged with implementing an already extensive curriculum, many are forced to make difficult decisions about which topics to cover; consequently, some teachers may leave probability topics on the "cutting room floor."

Principles and Standards advocates spanning the grades with the study of data analysis rather than reserving it for the middle grades and secondary school (NCTM 2000, 48). Nevertheless, the limited attention given to probability in *Principles and Standards,* particularly in prekindergarten through grade 5, may lead some teachers to assume that little or no attention is warranted. *Principles and Standards,* however, is clear about the need to teach probability in elementary school. Opportunities to learn probability concepts can be presented even if the emphasis on probability is less extensive than that placed on other strands, such as number and geometry. Moreover, the interconnectedness of the school mathematics curriculum offers abundant opportunities for learning probability concepts without sacrificing valuable instructional days.

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