PROBABILITY on a Budget

WILLIAM A. EWBANK AND JOHN L. GINTHER

Teachers do not need expensive equipment to present intriguing activities on probability in the classroom. Moreover, as *Principles and Standards for School Mathematics* (NCTM 2000) advocates, both probability and problem solving should feature prominently in the mathematics teacher's repertoire. This article describes a lesson using regular dice and specially marked dice to explore some of the concepts of probability. Students need to be aware that dice used in these experiments should be as uniform in construction as possible for gathering reliable data. One way to test the fairness of *any* die would be to roll it 100 times and see if each of the six sides comes up roughly 1/6 of the time.

WILLIAM EWBANK *teaches at Taylor University, Upland, IN 46989-1001. He established a mathematics lab in 1972 and found that simple games could be useful for teaching mathematics concepts and offering hands-on fun for students.* JOHN GINTHER, *john.ginther@emich.edu, teaches mathematics at Eastern Michigan University, Ypsilanti, MI 48197.*

Beginning the Lesson

BEFORE THE MAIN ACTIVITY, WE FIRST ASCERtain that the class has some idea that a die can land on 1, 2, 3, 4, 5, or 6; that is, P(*of each outcome*) = 1/6. Then we hand out a pair of special trick dice to one student and a regular pair to another. The trick dice are homemade (the pattern is given in the **Appendix**) and have faces numbered 5, 5, 5, 5, 5, 5 and 2, 2, 2, 6, 6, 6. Another method to create trick dice easily is by converting regulation dice or blank cubes by affixing adhesive numerals or dots to them.

We then discuss that total we are more likely to get if we roll two regular dice at once. From the sample space (see **fig. 1**), all the students can see that 7 is the most likely total, with P(7) = 1/6, and that the possible totals range from 2 to 12, yielding eleven possible totals.

The trick dice, however, can give totals of only 7 or 11, as can be seen from the sample space (see **fig. 2**). When we ask the two students to roll their pairs of dice and call out the totals, the trick dice may seem to fit our theory better than the genuine ones, until the constant repetition of 7 and 11 arouses interest. Often the student rolling the trick dice has not bothered to check the face numbers and is at first as mystified as anyone.

Having broken the ice with this trick, we then embark on a "what if?" exploration. Blank dice can be made using the pattern from the **Appendix.** Alternative dice can be made by using gummed blank stickers to cover the faces and writing the required numbers on them or by using wooden cubes. Such dice give students the chance to try the activity with different sets of numbers.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Fig. 1 Sample space showing possible totals for two regular dice rolled simultaneously

	5	5	5	5	5	5
2	7	7	7	7	7	7
2	7	7	7	7	7	7
2	7	7	7	7	7	7
6	11	11	11	11	11	11
6	11	11	11	11	11	11
6	11	11	11	11	11	11

Fig. 2 Sample space showing possible totals for a pair of trick dice rolled simultaneously

Exploring Probability

ONE INTERESTING TASK IS TO HAVE STUDENTS create all the possible dice with face totals of 21. Repetitions of numbers are allowed in this task. An example of such a die would have faces of 1, 2, 4, 4, 5, 5. Our list of possible number combinations for the faces of these dice appears in **figure 3.** (Notice that the letter I has not been used so as to avoid any misinterpretation by students.) The next task would be to find the "top die," if any, by rolling dice in pairs and scoring points for the higher number rolled.

Can these unconventional dice be "seeded," as contestants are in a knockout tennis tournament? Have two students play a "singles" match, each rolling a die. If A's number exceeds B's, A wins a point. If B's number exceeds A's, B wins a point. If the numbers match, the game is a draw. With regular dice, these contests would be fair, but are they fair when students use the special dice? If the match ends in a draw, we mark the sample space with DR. The number of DRs is crucial to the investigation.

For example, in **figure 4,** we can see that rolling dice C and U resulted in four draws and that rolling dice D and J resulted in five draws. If the number of draws is odd, one die must have more wins than the other because 36, the size of the sample space, minus an odd number gives an odd number, and the number of wins could not match the number of losses yet total an odd number. We can say, then, that J is more likely to beat D even though their face totals are the same because J's probability of winning is 17/36 and D's is 14/36.

A—{1, 1, 1, 6, 6, 6}
B—{1, 1, 2, 5, 6, 6}
C—{1, 2, 2, 4, 6, 6}
D—{2, 2, 2, 3, 6, 6}
E—{1, 1, 3, 5, 5, 6}
F—{1, 1, 4, 4, 5, 6}
G—{1, 2, 2, 5, 5, 6}
H—{2, 2, 3, 3, 5, 6}
J—{1, 3, 3, 3, 5, 6}
K—{1, 3, 3, 4, 4, 6}
L—{2, 2, 3, 4, 4, 6}
M—{1, 1, 4, 5, 5, 5}
N—{1, 2, 3, 5, 5, 5}
O—{2, 2, 2, 5, 5, 5}
P—{2, 2, 3, 4, 5, 5}
Q—{1, 2, 4, 4, 5, 5}
R—{2, 3, 3, 3, 5, 5}
S—{3, 3, 3, 3, 4, 5}
T—{2, 3, 3, 4, 4, 5}
U—{1, 3, 4, 4, 4, 5}
V—{1, 4, 4, 4, 4, 4}
W—{2, 3, 4, 4, 4, 4}
X—{3, 3, 3, 4, 4, 4}
Y—{3, 3, 3, 3, 3, 6}

Fig. 3 Possible face numbers that total 21 on the special dice. The numbers are based on the set {1, 2, 3, 4, 5, 6}.

This exploration leads to the next question: Could all the possible dice be seeded, that is, ranked as best, next best, and so on? By drawing out a huge sample space, made of 276 smaller sample spaces, we could come up with a solution. Using the twenty-four different dice in our list (see **fig. 3**), we will create the 276 smaller sample spaces. Note that a die cannot compete against itself and that die A versus die B is equivalent to die B versus die A. Die A compared with B through Y gives 23 sample spaces; die B compared with C through Y gives 22 more, and so on, for a total of 276. In a class of twenty-three students, if each student worked out twelve sample spaces, the chart could be completed.

What information does this huge sample space give? From **figure 4,** in which rolls of dice C and U resulted in four draws, we see that each die had sixteen wins. If the number of draws is an even number, are the two dice evenly matched? In the example of dice C and U, they are, but only scrutiny of all entries in the giant sample space will answer this question completely.

Can we look at any two given dice and tell whether they are evenly matched? We discovered that we can be certain that two dice are unevenly matched by a simple test, but if they fail the test, we cannot be certain. The clue lies in the concept of odd and even. For example, if die C has 1, 2, 2, 4, 6, 6 on its faces and if die G has 1, 2, 2, 5, 5, 6 on its faces, we calculate the number of possibilities for draws as follows:

The 1's match, yielding one DR	1
The 2's match, yielding four DRs	4
The 6's match, yielding two DRs	<u>2</u>
	7

		C					
		1	2	2	4	6	6
U	1	DR	C	C	C	C	C
	3	U	U	U	C	C	C
	4	U	U	U	DR	C	C
	4	U	U	U	DR	C	C
	4	U	U	U	DR	C	C
	5	U	U	U	U	C	C

(a) Distribution of wins between special dice C and U

		D					
		2	2	2	3	6	6
J	1	D	D	D	D	D	D
	3	J	J	J	DR	D	D
	3	J	J	J	DR	D	D
	3	J	J	J	DR	D	D
	5	J	J	J	J	D	D
	6	J	J	J	J	DR	DR

(b) Distribution of wins between special dice D and J

Fig. 4 Sample spaces showing outcomes of investigation of two pairs of special dice

The number of possibilities for draws is 7, an odd number, which means that dice C and G could not be evenly matched. Students can usually determine whether the total possibilities for draws will be odd or even by inspecting the faces of the dice.

No single die from our list seems likely to beat all the others. That is, the dice cannot theoretically be placed in a "seeded" order.

Whatever the outcome of this intense theoretical speculation, each result can be compared with the experimental result of actually rolling the dice. This assignment can be done as homework, especially if the students have made their own dice. Of course, homemade dice will not roll as evenly as factory-made ones.

Closing Thoughts

IN TENNIS, OR ANY OTHER SINGLES GAME, SUCH as chess or squash, if player A is likely to beat player B and if player B is likely to beat player C, everyone will agree that player A is likely to beat player C. Symbolically, this idea reads as follows:

(A beats B, and B beats C) implies that (A beats C).

With special dice, however, this scenario does not necessarily hold true. Here is an example using four dice, in which die I has faces of 3, 3, 3, 3, 3, 3; die II has faces of 1, 1, 1, 5, 5, 5; die III has faces of 0, 0, 4, 4, 4, 4; and die IV has faces of 2, 2, 2, 2, 6, 6. Students could be invited to verify the strange result.

$$\begin{tikzcd} A \arrow[r] & B \arrow[d] \\ D \arrow[u] & C \arrow[l] \end{tikzcd}$$

(In this diagram, $\longrightarrow$ = "beats.")

In playing the simple rolling game in which the larger face number wins one point, a player could let the opponent choose her or his die first, then could always choose one that is likely to beat it! The probabilities for contests involving dice I and III and dice II and IV are left as an exercise for the reader. What other sets of four dice would have this curious characteristic? The number of sets would appear to be infinite.

From the humble regular die, we strayed into unexplored territory. Some might say that we opened a can of worms, but our students took the bait!

Reference

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

APPENDIX

Paper Cubes

Objective

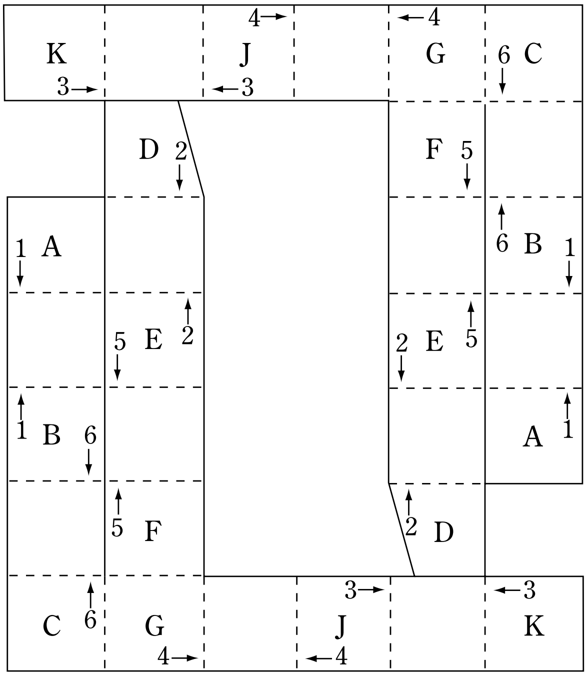
To form a sturdy paper cube by folding. The resulting cube need not be glued and can be marked and used for dice games.

Materials

1. Pattern for cubes (see the pattern at right)
2. Scissors
3. Paper glue (optional)

Preparation

1. Cut the pattern along the heavy, solid lines. The pattern supplies a plan for making two identical cubes.
2. Fold back and crease each dotted line on one of the two cube nets that you have created from the pattern. (Note: Precision in cutting and creasing is important!)
3. Form the cube by putting one square over another, as follows:



Pattern for Making Two Cubes

Instructions for Folding	Instructions for Gluing (A glued cube is somewhat stronger than one that is folded only.)
1. Place square 6 over square F.	Do not glue.
2. Place square 4 over square B.	Glue.
3. Place square 5 over square J.	Glue.
4. Place square E over empty space.	
5. Place square A over square C.	Glue twice.
6. Place square 3 over square A.	Glue.
7. Place square K over square G.	Glue.
8. Place square 2 over square K.	Do not glue.
9. Place square D under square 6.	Glue twice, including both sides of D.