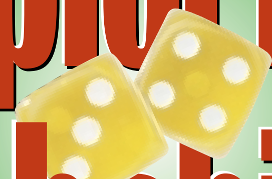
# Exploring Probability



## through an Evens-Odds Dice Game

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MIDDLE-GRADES STUDENTS SHOULD HAVE opportunities to experiment actively with situations that model probability, including "making hypotheses, testing conjectures, and refining their theories on the basis of new information" (NCTM 1989, 111). These experiences should include a discussion of theoretical probabilities, where appropriate, and the use of correct mathematical terminology and expressions.

Exploring probabilistic thinking not only exercises students' general reasoning abilities but also helps build a foundation for more informed decision making in everyday life. Students, like adults, regularly use their knowledge of probabilities to make choices about important issues, such as health and safety, and to determine strategies in some recreational pastimes, such as sports and games. In this article, we describe a dice game that can be used by students as a basis for exploring mathematical probabilities and making decisions while they also exercise skills in multiplication, pattern identification, proportional thinking, and communication. As we observed and listened to the sixth graders participating in the activity, we gained some interesting insights into student thinking at this age level.

## Game Overview

STUDENTS PLAY THE GAME IN PAIRS, ALTHOUGH groups of three or four would also work. On her or his own game board (see **fig. 1**), each student divides a given number of chips between the two categories of "even" and "odd." Students take turns rolling two dice, finding the product of the values on the faces, and removing—only for their own roll—an "even" chip for an even product and an "odd" chip for an odd product. The first to remove all chips for both categories wins. Rolls that come up for a category for which all chips have already been cleared result in a lost turn. Students get an equal number of turns, so whoever rolls the dice second at the start of a two-player game may roll last.

## Playing with Tetrahedral Dice

TO BEGIN PLAYING THE GAME, STUDENTS WERE given a pair of tetrahedral, or four-faced, dice; a game board; and sixteen chips. Dice with a greater number of faces could be used. Students were divided into four-member discussion groups. Each student had a game board, and each pair of students had two four-faced dice. Students were instructed how to read these dice. After playing instructions were given, students were asked to take a few minutes to discuss in their groups the strategies that they might consider for dividing their chips into the "even" or "odd" categories on their game boards. The group then split into two pairs to play the game. Each student decided how to place her or his chips on the game board.

Unsurprisingly, most sixth graders who participated in this lesson evenly split their chips into two groups of eight for the first game. Students played the game once then returned to their groups of four to discuss the effectiveness of their strategies and ways to revise them for replaying the game. Most noticed that more even than odd products had appeared. However, they could not seem to determine why. They decided, though, to place more chips on "even" for the next game. In the next game, the students' strategies were more varied; most students placed from eleven to fifteen of their sixteen chips on "even," with the majority choosing fourteen and fifteen.

After this second game, students again discussed with their group the success of their strategies and possible revisions, and they were encouraged to search for systematic or organized ways to

| EVEN | ODD |
|---|---|
| | |

**Fig. 1 An "evens-odds" game board**

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determine how they might place the chips to increase their chances of winning. Still, they knew only that more evens than odds had appeared without seeing a mathematical justification. We were pleased, however, that one girl described using proportional reasoning. She noticed, "One out of every five times I get an odd. That's why I put three chips on odd because that would be out of fifteen." Although the expected proportion is really one out of four, her thought process was correct. Asked how many chips they would place in each category if they were to play the game again, more than half the class said that they would place from twelve to fourteen chips on "even."

We asked students to return to their pairs to play the game a third time by using twelve chips but without discussing strategies with their group. Two-thirds of the students chose an 8-4, 9-3, or 10-2 even-odd split, with half of these students opting for 9-3. For this game, we asked students to record their rolls, including missed turns, by placing pairs of factors in an even- or odd-products column. **Figure 2** shows one student's recorded rolls.

Using their game data, students discussed their results for this game in small groups. We then discussed their findings as a class. Students said that the chances of getting odd products were fewer because only 1's and 3's appeared in that column on their recordkeeping sheets. This conclusion was the extent of any rule or pattern they detected. We listed on the chalkboard possible pairs of factors that would yield even or odd products (see **fig. 3**). We then generated these possible outcomes—the sample space—systematically, as the organized list in **figure 4** shows. Students might also make a tree diagram to illustrate possible pairs.

| Even Products | | | Odd Products |
|---|---|---|---|
| 3, 4 | 4, 4 | 2, 3 | 1, 1 |
| 1, 4 | 2, 2 | 4, 3 | 3, 1 |
| 2, 4 | 3, 2 | 4, 2 | 3, 3 |
| 4, 1 | 2, 1 | 1, 2 | 1, 3 |

**Fig. 3 Randomly generated list of pairs of factors that yield odd or even products for rolled "factors" on two tetrahedral dice**

Students discovered that even products should, theoretically, appear three-fourths of the time and odd products, one-fourth of the time. The lists helped make apparent that even-even, even-odd, and odd-even combinations result in even products and account for three-fourths of all possible products, whereas only odd-odd combinations result in odd products, or one-fourth of all possible products. The best strategy for placing sixteen chips, then, would be to put twelve, or three-fourths, in the "even" category and four, or one-fourth, in the "odd." For twelve chips, nine chips would be put in

| Outcomes of Dice Rolls | | | | | | |
|---|---|---|---|---|---|---|
| Even Products | | | | Odd Products | | |
| Factor | Factor | Product | | Factor | Factor | Product |
| | | | | | | |
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**Fig. 2 A student's recording sheet for dice-roll products**

"even" and three in "odd." For any one roll of the dice, an *event,* the probability then is 3/4 for getting an even product and 1/4 for getting an odd product, which represents in each example the ratio of favorable outcomes to total outcomes. As an extension, the class can also examine other probabilities related to the game, such as the probability that a particular student who has only two "even" chips remaining will finish the game in her or his next two rolls of the dice. The probability of getting even products for two consecutive rolls is

$$\frac{3}{4} \times \frac{3}{4} = \frac{9}{16};$$

for two consecutive odd products, the probability is

$$\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}.$$

These ideas should be discussed and worked through carefully with students, and students should understand that ratios obtained by experimentation, as in the evens-odds game, are approximations to theoretical probabilities. Experience in playing the game can help students improve their strategies, but a mathematical analysis of probabilities, when possible, is the best foundation for decision making. Another important concept is that estimates of probabilities made from ratios obtained by experiment should improve as the number of trials increases. In this example, probability estimates for even and odd products tend to approach 3/4 and 1/4, respectively, as the number of rolls increases.

After students had played the game with tetrahedral dice, we had them play it with standard dice. Different groups of students started with different numbers of chips—fourteen, sixteen, twenty-four, or thirty-six—only the last three of which can be divided evenly into three-fourths and one-fourths proportions. We made this distribution not only to save time from having all students experiment with these initial numbers but also to enrich our understanding of students' thinking and to foster better class discussion. Interestingly, more than one-third of the students reverted to using the same number of chips, such as twelve and twelve, in both categories. This lack of generalization of the pattern we had discussed after the earlier games with the tetrahedral dice shows that students continue to need much practice thinking about what they are doing and applying it to new situations by identifying like and unlike features of source (old) and target (new) situations and what effects, if any, the differences might have.

At various points in these games, we asked students who of their pair they believed was currently winning. We wanted to see whether students would reason on the basis of whose chips more nearly ap-

| | | | |
|---|---|---|---|
| (1) 1, 1 | (2) 2, 1 | (3) 3, 1 | (4) 4, 1 |
| (2) 1, 2 | (4) 2, 2 | (6) 3, 2 | (8) 4, 2 |
| (3) 1, 3 | (6) 2, 3 | (9) 3, 3 | (12) 4, 3 |
| (4) 1, 4 | (8) 2, 4 | (12) 3, 4 | (16) 4, 4 |

**Fig. 4 Possible outcomes, or sample space, for two tetrahedral dice; products in parentheses**

proximated the three-fourths (even)-to-one-fourth (odd) proportion that represents the theoretical probability for this activity. Students did not offer such an explanation. They were apt to respond, “Me, because I have fewer chips” or “Me, because I still have one on odd and she doesn’t,” meaning, in the latter statement, a reduced chance of having to skip a turn. Teachers can ask this question informally of pairs while moving about the room or pose the question to the entire class during a time-out for all pairs to discuss the question briefly.

## Students’ Response to the Game

OUR STUDENTS LOVED PLAYING THE GAME. THEY said that they enjoyed the hands-on aspect and the challenge posed by thinking about better strategies and expressing opinions about their ideas. Asked what they gained or learned, students highlighted three things:

- Practice with multiplication skills
- Finding that more even than odd products of possible number-pair combinations exist
- The pattern (E = even, O = odd): $E \times E = E$; $E \times O = E$; $O \times E = E$; $O \times O = O$

## Game Extensions and Variations

STUDENTS CAN ALSO PLAY THE GAME WITH DICE of eight, ten, twelve, or twenty faces, which are common in many sets of polyhedral dice. Since ten-faced dice are usually numbered 0–9, 0 can represent 10. These dice can be mixed and matched. Note that the pattern found in the original version of the game still holds here: An equal number of even and odd numbers on each die, that is, an even number of numbers, yields possible products that are three-fourths even and one-fourth odd. Students could also play the game with three dice, or at least hypothesize the results. In this situation, only one of eight possible combinations—odd on all three dice—yields an odd product. Students can look for patterns as the number of dice increases. A geometric progression results in which, for example, the probability of getting an odd product can be shown as $1/2^n$ where $n$ = number of dice; the probability of attaining an odd product decreases as the number of dice increases. Calculators might be appropriate for some of these game variations.

One hypothetical question to pose to students is whether using a die with an odd number of faces would change the three-fourths-to-one-fourth proportion of the “evens-odds” game. It would. For example, with a four-faced and a five-faced die, assuming that both are numbered consecutively beginning with 1, the proportion would change to seven-tenths-even (fourteen of twenty products)–to–three-tenths-odd (six of twenty products).

A game variation could be to play the game using the two categories of “doubles” and “nondoubles.” For two tetrahedral dice, doubles would occur one-fourth of the time and nondoubles, three-fourths of the time; for two standard dice, doubles would occur one-sixth of the time and nondoubles, five-sixths of the time; and so on. This pattern holds for two identical dice that have an even number of faces; a spin-off could be to use two dice with different numbers of faces.

## Closing Thoughts

MATHEMATICAL EXPERIENCES THAT ENCOURAGE students’ reasoning are most valuable. They are even more so when they require probabilistic thinking that leads to decision making. In this article, we described one such experience in which students used a hands-on game format to exercise these and several other mathematical abilities.

## Reference

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

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