Meaningful Mathematical Representations and Early Algebraic Reasoning

Both oral and written communication play an important role in teaching and learning mathematics (NCTM 2000). Students and teachers exchange ideas about their understanding of, and thinking about, mathematics by communicating with one another. An important part of this communication process is the choice of symbols used to represent that thinking (Hiebert 1989). The process of representing mathematical ideas using symbols and expressions should begin at the earliest stages of mathematics instruction and appear in the context of ideas to which young students can relate (Carey 1992).

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This article illustrates one way that a children's book can be used to develop young students' abilities to make sense of mathematical representations. We selected this mathematical task to establish early in the primary school curriculum a better understanding of what the equals symbol represents. This experience should give students a richer understanding of equations that appear later in the curriculum and act as a precursor to the development of algebraic thinking.

We hope that in reading this article, teachers will be motivated to reflect on the ways that mathematics is used in their communities and how these uses are culturally influenced. With this understanding, teachers can then create connections for students between the mathematical concepts to be taught and the uses of these concepts in the world outside the classroom.


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Background

Thompson and others (1994) emphasize the importance of a teacher's having a "conceptual orientation." An outcome of this orientation is found in students' explanations of their solution strategies when they attach meaning to numbers and arithmetic expressions and when they look at relationships in the context of a problem. This result is in contrast with a "calculational orientation," which is seen when students focus on procedures to obtain an answer.

Herscovics and Kieran (1980) note that students have a strong tendency to use the equals sign in a very narrow sense, namely, to indicate only the results of an operation. For these students, the equals sign means "perform an operation and show the result." For example, these students would maintain that 3 + 4 should be followed by an equals sign to indicate the need to add to arrive at 7. These students seem unwilling to accept 3 + 4 as representing something other than 7. Students who think about the equals sign in this manner do not fully appreciate that this sign represents a relationship of equality between two quantities. As Herscovics and Kieran (1980) point out, this type of understanding creates additional obstacles in the more formal study of equations that occurs later in the curriculum. Furthermore, students' desire to find a sum for such an expression creates a cognitive obstacle when they first encounter such representations as $3x + 5$ or $a^2 + a^3$; they believe they need to complete the calculation to further simplify the expression.

When young students recognize that the equals sign describes the relationship of equality between two quantities, we found that they begin to focus their reasoning on the quantities and operations, not just the numbers. This increased level of abstraction is instrumental in developing prealgebraic thinking.

A Worthwhile Task to Establish Meaning for Mathematical Representations

We chose *How Many Snails?* (Giganti 1988) to use in discussing representations in a first-grade class of twenty-eight students. This counting book can also help students realize that numbers and arithmetic expressions serve as representations. One of the authors of this article, an experienced classroom teacher, led the discussion with the first-grade students, and the other observed and offered suggestions to contribute to the teacher-student discourse. The session was videotaped and transcribed, providing an accurate record of the dialogue, which has been condensed for this article, and enabling us to easily identify major points in the discussions.

The teacher began by showing the first two pages of the book, which have a number of clouds on each page (see **fig. 1**).

Teacher. If I were talking about how many clouds are big and how many clouds are small, what would I write down to talk about that? Sarah?

Sarah. Um, there's two big ones and eight little ones.

Teacher. Do you agree?

Jeffrey. No, there are three big ones and five little ones.

To encourage listening for understanding, the teacher asked Justine, who had agreed with Jeffrey's statement, to repeat what Jeffrey said. The teacher then continued to focus on the counting and what the counting represented. Once the students agreed on the number of clouds, the teacher moved to a symbolic representation of the students' counting.

Teacher. He said three big clouds and five little clouds: 1, 2, 3 big ones [counting the big clouds]; 1, 2, 3, 4, 5 little ones [counting the little clouds]. [Writes 3 and 5 on the board.] Now, what would I write mathematically to show there's three and there's five? Thomas?

Thomas. A plus.

Teacher. Do you agree? Angelina?

Angelina. Yes.

Teacher. I am going to put that [writes a plus between the 3 and the 5] right here, OK. What does the 3 represent? Zach?

Zach. Three big clouds.

Teacher. The number of big clouds. What does the 5 represent? Colleen?

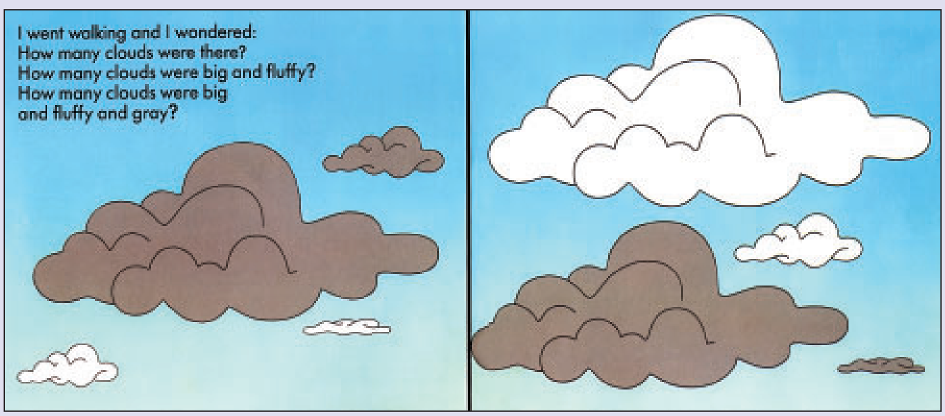
Colleen. Five little clouds.

Note that the teacher has reinforced what the numbers 3 and 5 represent in the arithmetic expression 3 + 5. Also, she referred to the 3 and 5 as numbers representing quantities, in this example, numbers of clouds. The students must understand that 3 and 5 are numbers that represent how many objects. At this stage, however, in first grade, the teacher should probably just state that fact rather than correct the student because the focus of the lesson is representation. In this class, the teacher then moved on to the meaning of the whole expression 3 + 5.

Teacher. What are you going to say if we ask what does that number [pointing to the expression

FIGURE 1

The first two pages of *How Many Snails?* (Giganti 1988)



3 + 5] represent together? Don't tell me 8. What does this talk about? Zach?

Zach. The clouds.

Teacher. What clouds?

Zach. Large and small.

Teacher. It talks about all of the clouds, doesn't it?

The emphasis of instruction is on what is represented by 3 + 5; the sum represents the number of clouds, not simply 8. Coming to this understanding is an essential first step in establishing that the equals sign means more than "perform an operation and show the result." Now another representation for the number of clouds can be made.

Teacher. OK, now I want to look at the clouds and I want to talk about the number of gray clouds and the number of white clouds. What can you tell me about those numbers? Nicholas?

Nicholas. They are both four.

Next, Jessica told the teacher that four plus four represents the number of gray clouds and the number of white clouds, and the teacher wrote 4 + 4 on the board. She continued:

Teacher. What can you tell me about this number [3 + 5] and this number [4 + 4]? Aaron?

Aaron. They both equal 8.

Teacher. OK, they both equal 8. What else about this number and this number? Angelina?

Angelina. It's 3, and that means it's going backwards because it is 4 and then it's 3 and the other one is 4 and then it is 5.

At this point in the discussion, some students were focusing on the numbers and the numerical relationships that they observed in the two arithmetic expressions, not on what was represented by the expressions. The teacher continued to probe, asking what else they observed.

Teacher. What else can you tell me?

Greg. They both are all of the clouds.

Teacher. This is all of the clouds [3 + 5] and this is all of the clouds [4 + 4], and we know that's the same number. What could I put here [pointing to the space between the two expressions] to say that this is the same as this? What symbol in mathematics says this is the same as this? Daria?

Daria. Equals.

The teacher wrote 3 + 5 = 4 + 4 on the board. She took time to have the students explain what each number and each arithmetic expression represented to give them practice in communicating their understanding of the representation.

These students were helped to realize and understand that even the numbers in a simple sum can be thought of as representing the number of objects in some situation. They also learned that not carrying out an addition operation whenever they see a plus sign is acceptable. This experience will help prepare the students to accept the idea that some algebraic expressions, such as $3y + 2$, cannot be further simplified by adding the terms. They have used the equals sign as a way to represent the equality between two quantities, not just as a symbol signifying the instruction to perform an operation.

Throughout this discussion, we observed a change in the thinking and reasoning of the students. They moved from counting and reporting numbers to realizing that the number of clouds is an invariant that can be represented as a sum in more than one way. This form of more generalized reasoning in arithmetic is representative of prealgebraic thinking.

Next, the teacher turned to a two-page layout showing yellow, pink, and white flowers with black, yellow, or orange centers. Because of the use of color in this picture, many more different forms of representations can be discussed than with the clouds. The teacher asked the students first to describe the colors of the flowers, then to count how many of each flower was in the picture. They found nine yellow flowers, four pink flowers, and two white flowers, leading to the arithmetic representation 9 + 4 + 2. The teacher wrote this representation on the board.

The teacher then asked a series of questions about what was represented by each of the numbers 9, 4, and 2. After looking at the colors of the centers of the flowers, the class came up with a second representation, 9 + 3 + 3, representing the number of flowers with black, yellow, and orange centers, respectively. By referring to each arithmetic expression as representing the total number of flowers on the two pages, the students recognized the relationship of equality between the two representations and described it as 9 + 4 + 2 = 9 + 3 + 3. This activity reinforced the many themes found in the work with the clouds.

The teacher could also write the expression 8 + 2 + 1 + 1 + 1 + 1 + 1 on the board and ask students to identify characteristics of the flowers that could be used to generate this representation, which considers both color of flower and color of center. Note that this expression has more than three terms. Thus, students would see the equality of one expression that has three terms and another that has seven. This representation further enhances the students' concept of what the equals sign means.

In addition, the teacher could ask the students to describe different ways to count the number of flowers. This question allows students to identify the characteristics that they would use to count. These kinds of explorations of relationships for number facts give students experience with the type of generalizing that is frequently called prealgebraic thinking. Furthermore, these formulations of the problem are instrumental in helping students extend their ideas of representation and in assessing how well students understand various representations.

Possible Extensions

Additional worthwhile tasks can be used to develop other aspects of prealgebraic thinking, including the following:

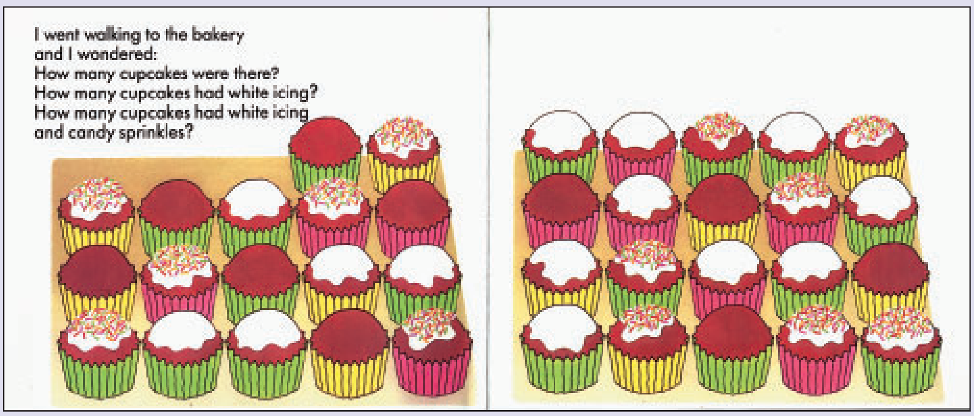
- *Commutativity of addition*. In the discussion of the number of clouds, the students could also have been asked to count the little clouds first, then the big ones, resulting in the equation 3 + 5 = 5 + 3. From these two representations, students would begin to build an understanding of the commutative property of addition. This focus on the properties of operations as the objects of study is another component in the development of prealgebraic thinking.
- *Equations with an unknown*. Using the two-page layout of fish swimming in *How Many Snails?*, the teacher could show the students the first page, which shows six fish, and ask, “How many fish are on the second page if both pages have a total of fourteen fish?” Using arithmetic representations, this extension establishes informal groundwork for the concept of equations. This foundation could be extended by adding the condition that the second page shows three fish with their mouths open, then asking how many other fish are on the second page. The teacher might ask, “How many fish on the second page could have their mouths open, and how many could have their mouths closed?” This question introduces the process of finding whole-number solutions to an equation with two variables. All these questions depend on students’ making the connections between the number of objects on the page and their corresponding numerical representations.
- *Expressions using addition and multiplication*. One two-page layout of *How Many Snails?* shows rows of cupcakes (see **fig. 2**). Using the arrangements of the cupcakes, teachers can guide students to think of multiplication as repeated addition or as arrays. For example, the number of cupcakes on the second page could be represented by 4 × 5 or as 5 + 5 + 5 + 5 because the picture shows four rows of five cupcakes each. The first page shows three rows, each with five cupcakes, and two cupcakes in a fourth row, resulting in the representation 3 × 5 + 2 for the number of cupcakes on that page. Encouraging multiple representations with counting results in expressions that include both addition and multiplication. In this task, the same situation can be described by different operations, another aspect of prealgebraic thinking.

Summary

Students’ use of representations to organize their thinking and to communicate ideas is essential in developing their problem-solving abilities, understanding of mathematical ideas, and prealgebraic thinking. Students at all grade levels should be given the opportunity to create and use representations. Our vignette suggests how one phase of this development can begin in first grade. Working with

FIGURE 2

A two-page layout from *How Many Snails?* (Giganti 1988), showing rows of cupcakes



a children's counting book, we demonstrated how arithmetic expressions can be used as vehicles to develop students' understanding of representations. These representations were used to model a physical situation found in a motivational context for students, namely, an interesting and colorful book. Moreover, the students were given the chance to develop their own representations because they were allowed to choose what characteristics they would count. These students gained experience in working with both representations and other important mathematical ideas.

Most important, students learned that the equals sign means more than "perform an operation and show the results." It is a symbol that represents the relationship of equality between two quantities. The activities in this lesson helped students become comfortable with expressions that are not simplified, preparing them to understand that not all algebraic expressions can be simplified as a single term. In addition, the students' informal experiences with equations containing unknowns will prepare the way for more formal work with equations. They can begin to see equations as representations of equality between two quantities, not just as problems to be solved.

Throughout these discussions between teacher and students, we saw a shift in the students' thinking from focusing on just numbers to considering more general relationships between numbers used to represent quantities and operations. This shift illustrates the prealgebraic thinking that must be developed in primary-age students. Overall, the students in one first-grade classroom gained extensive experience in working with representations and other important mathematical ideas.

References

Carey, Deborah A. "Students' Use of Symbols." *Arithmetic Teacher* 2 (November 1992): 184–86.

Giganti, Paul, Jr. *How Many Snails?* Illus. by Donald Crews. New York: William Morrow & Co., 1988.

Herscovics, Nicolas, and Carolyn Kieran. "Constructing Meaning for the Concept of Equation." *Mathematics Teacher* 73 (November 1980): 572–80.

Hiebert, James. "The Struggle to Link Written Symbols with Understandings: An Update." *Arithmetic Teacher* 36 (March 1989): 38–44.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, Va.: NCTM, 2000.

Thompson, Alba G., Randall A. Philipp, Patrick W. Thompson, and Barbara A. Boyd. "Calculational and Conceptual Orientations in Teaching Mathematics." In *Professional Development for Teachers of Mathematics,* 1994 Yearbook of the National Council of Teachers of Mathematics (NCTM), edited by Douglas B. Aichele and Arthur F. Coxford, 79–92. Reston, Va.: NCTM, 1994. ▲