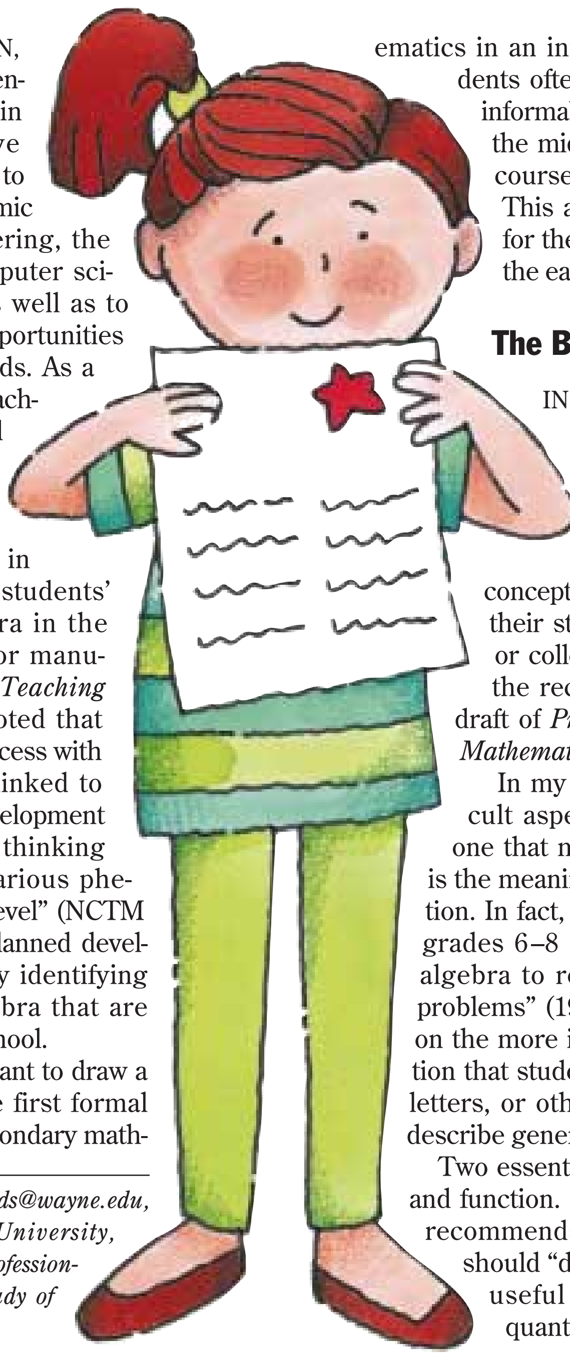
# Some "Big Ideas" of

# *in the*

THOMAS G. EDWARDS



WITHOUT QUESTION, mathematics in general, and algebra in particular, have served as "gatekeepers" to the study of other academic fields, such as engineering, the physical sciences, computer science, and medicine, as well as to increased vocational opportunities in technical support fields. As a result, middle school teachers have felt increased pressure both to teach algebraic concepts directly and to develop mathematical concepts in ways that will support students' formal study of algebra in the future. A recent call for manuscripts in *Mathematics Teaching in the Middle School* noted that "the rate of students' success with this subject has been linked to the careful, planned development of algebra as a way of thinking about and modeling various phenomena at every grade level" (NCTM 1999). Such a careful, planned development requires clearly identifying the "big ideas" of algebra that are appropriate to middle school.

Before beginning, I want to draw a distinction between the first formal course in algebra (or secondary mathematics in an integrated curriculum), which students often study in grade 8, and the more informal study of algebra-related topics in the middle grades before the first formal course in secondary-level mathematics. This article discusses a set of big ideas for the more informal study of algebra in the early middle school years.

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## The Big Ideas

IN THINKING ABOUT WHAT MIGHT constitute the big ideas of algebra in the early middle grades, I have drawn from my own professional experience, which has armed me with a short list of algebraic concepts that I hope my students bring to their study of algebra at the high school or college level. This list corresponds to the recommendations in the discussion draft of *Principles and Standards for School Mathematics* (NCTM 1998).

In my experience, one of the most difficult aspects of algebra for students, and one that most teachers seriously underrate, is the meaningful acquisition of algebraic notation. In fact, the NCTM states that students in grades 6–8 should be able to "use symbolic algebra to represent situations and to solve problems" (1998, p. 222). This statement builds on the more informal grades 3–5 recommendation that students should be able to use "boxes, letters, or other symbols to solve problems or describe general rules" (p. 164).

Two essential algebraic concepts are variable and function. In discussing variable, the NCTM recommends that all students in grades 3–5 should "develop the concept of variable as a useful tool for representing unknown quantities" (p. 164) and that all students in grades 6–8 should "develop a sound conceptual understanding . . . of variable" (p. 222). With respect to the concept of function, the recommendations state that all students in grades 3–5 should "identify and describe relationships between two quantities that vary together" (p. 163), and their counterparts in grades 6–8 should "represent a variety of relations and functions" (p. 221). At both levels, the recommendations suggest that students' formal work with function might derive from more informal work with numerical and spatial patterns.

The final item on my short list of big ideas is properties of numbers. Although such concepts may seem more arithmetic than algebraic, the ability to generalize arithmetic properties is often the lifeblood of algebra.

The following paragraphs discuss the algebraic concepts on my list in more detail and offer some pedagogical suggestions that derive from my own experience as an algebra teacher.

#### Algebraic notation

Sinclair (1990) reminded us that according to Piaget, many young children experience difficulty in learning mathematics because symbolic mathematical notation is introduced to them prematurely (p. 28). My work with secondary and college students suggests that many of their difficulties are also related to mathematical notation. In fact, any attempt to force students at any level to use notation for which they have not yet established meaningful referents could be termed "premature."

A number of good algebra manipulatives are currently available that might be used to support algebra instruction in the middle school. In addition, ordinary manipulative materials that are often found in middle schools can be substituted for algebra-specific manipulatives. **Figure 1** shows manipulatives designed to help students develop meaningful referents for many variable expressions that are commonly used in algebra. The use of such referents helps students avoid a number of common errors. For example, when students use a "putting together" model of addition, the common error shown in the equation $2x^2 + 3x^2 = 5x^4$ would no longer make sense to them, because the result of putting together two collections of "squares" must be another collection of squares. A common error in subtraction can also be avoided when students have access to manipulatives. Beginning algebra students who write $4x - x = 4$ are probably "taking away" the $x$ from the expression $4x$. This mistake suggests the lack of a meaningful referent for the symbol $x$. When students have experience using manipulative materials to model such subtraction equations, as shown in **figure 2,** they are far more likely to apply a "taking away" model of subtraction appropriately and realize that the result of the operation must be $3x$.

Unfortunately, the links that are required for understanding relationships between abstract symbols and their concrete referents in these previous examples do not materialize simply because students use manipulatives. Students typically need sufficient experience in both decoding concrete representations of algebraic expressions and constructing concrete models of such expressions. **Figure 3** shows examples of both kinds of experiences.

**Students' difficulties are related to mathematical notation**

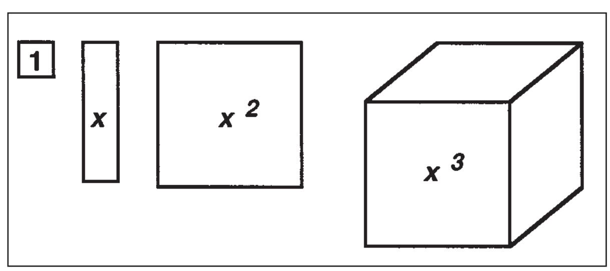



**Fig. 1 Manipulatives can provide meaningful referents for algebraic symbols.**

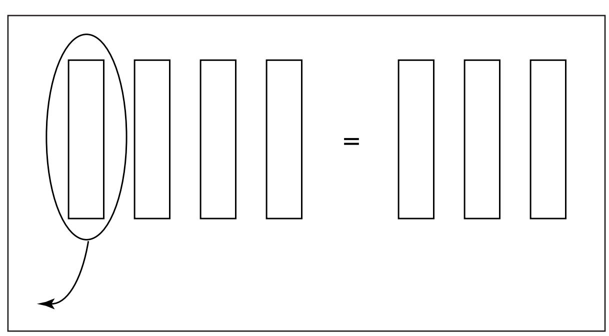
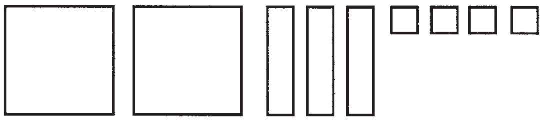



**Fig. 2 A "taking away" model of $4x - x$ using manipulatives**

1. Use the symbols of algebra to represent the expression pictured below.



2. Use manipulatives to build a model of $x^2 + 2x + 5$.

**Fig. 3 Students need a variety of experiences to develop links between concrete and symbolic representations.**

## Variable

The concept of variable is basic to algebraic representation and generalization, but Leitzel (1989) cautions that "the concept of variable is more sophisticated than we often recognize and frequently turns out to be the concept that blocks students' success in algebra" (p. 29). Beginning algebra students frequently face a fundamental difficulty in understanding the subtle distinction between a variable and an unknown in an equation. A focus on solving equations, which is often presented too early, can mask the true nature of the concept of variable. Finding solutions to first-degree equations clearly involves finding the one value for the variable, out of infinitely many possibilities, that makes the open sentence true; but the idea of "infinitely many possibilities" is often lost on beginning algebra students.

After all, if students are asked to solve $3x - 2 = 10$, then $x = 4$; in this sense, $x$ can hardly be thought of as varying. For the same problem, a different approach that might help students in constructing the concept of variable is to ask students to "find a value of $x$ for which $3x - 2 = 10$." Then ask, "Can you find a value of $x$ for which '$3x - 2$' equals 13? Equals 16? Equals 17?" Such sets of questions highlight the true nature of the variable $x$ while still providing experience in solving simple equations.

Another way to help students understand this concept is to begin the study of variable with formulas and tables of values. For example, once middle school students have been exposed to the formulas for the area and perimeter of a rectangle, they can work with these formulas to explore the effects of varying the length and width of the rectangle.

Tables of values can also be used to help students develop the concept of variable, and graphing calculators can greatly facilitate this work. Modern graphing calculators contain built-in table-generating features, but a potentially more powerful approach is to use the large-screen and automatic-recall features of the calculator first. For example, using the TI-82 or TI-83, students can enter the expression 3(4) – 2, press [ENTER], and press [2nd] [ENTER] to retrieve the previous entry. Then they can replace the 4 of the previous entry with a 5, press [ENTER] again, and continue this process. **Figure 4** shows what the screen might look like after several iterations of this process. Notice that several lines of the form 3( ) – 2 on the screen can easily lead into the general form $3x - 2$. To use the table-generating feature, first use the [Y=] editor to enter Y1 = 3x–2. Press [2nd] [TABLE] to display a table of values. When the table has been generated on the screen, press the [▶] and [▲] cursor keys to display the equation for Y1. **Figure 5** shows a table of values generated in this manner. This work with tables and formulas also serves as a natural transition to the concept of function.

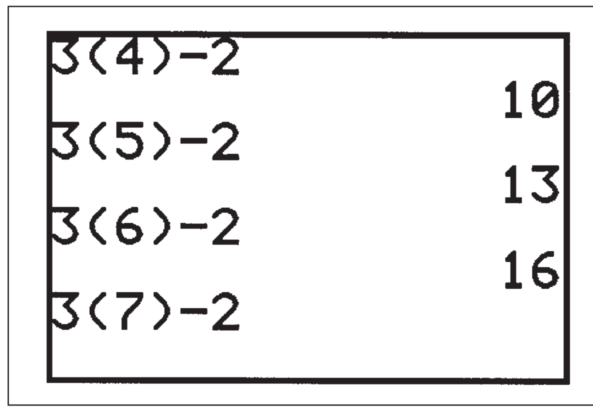



**Fig. 4 Building a table of values "by hand"**

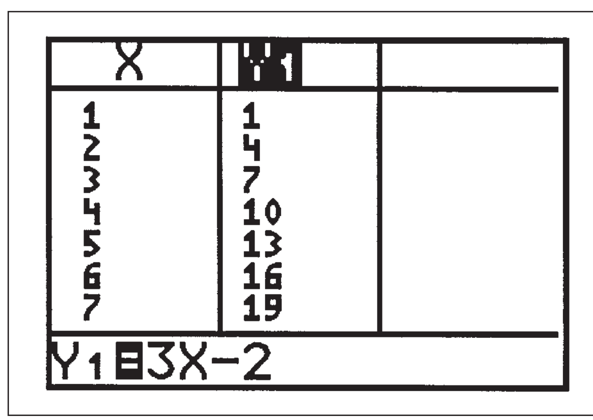



**Fig. 5 An automatically generated table showing the defining equation**

## Function

Thorpe (1989) believes that "functions should be taught to *all* students, because the concept of function is one of the most important of mathematical concepts" (pp. 12–13). As examples, he points out that one function assigns the amount of sales tax on a purchase and that another translates a rate of interest into monthly payments on loans. Thus, the study of function should have some intrinsic value for students.

A formal definition of function, including the concept of ordered pairs, is too abstract for, and probably unnecessary at, the middle-grades level. Rather, some experiences that introduce students to the idea that a function takes a specified input and returns a unique output make much more sense for this level. Two excellent activities that are appropriate in the middle grades are Guess My Rule and the Function Machine. These activities are described in Billstein, Liebskind, and Lott (1997, pp. 74–77). Each activity focuses on the input-output nature of functions, which is the most important property of functions that middle school students, and even high school students, need to understand.

Guess My Rule could be updated to Guess the Calculator's Rule if the teacher desires. Incorporating the calculator adds a connection to the symbolic representation of a function, because symbolic representations are used to communicate the rule to the calculator. For example, using a TI-82 or TI-83, the teacher could press Y= and key in a function, such as Y1= 2x + 3, before class. During class, students can suggest different values that can be entered for *x* from the home screen using the STO key. Then to display the value for Y1, students can press VARS ▶ to "Y-VARS." Next choose "1:Function" by pressing ENTER. Finally, press ENTER two more times to get the value for Y1. To get the display in **figure 6,** press 2 STO → X, T, θ, n ENTER. One powerful advantage of having students guess the calculator's rule instead of the teacher's is that the teacher is relieved of the burden of mentally computing the function value for each student-generated value of *x*. The teacher can then concentrate on the students' thinking, listening to determine, for example, whether the students are randomly proposing values of *x* or whether their proposed values have some pattern.

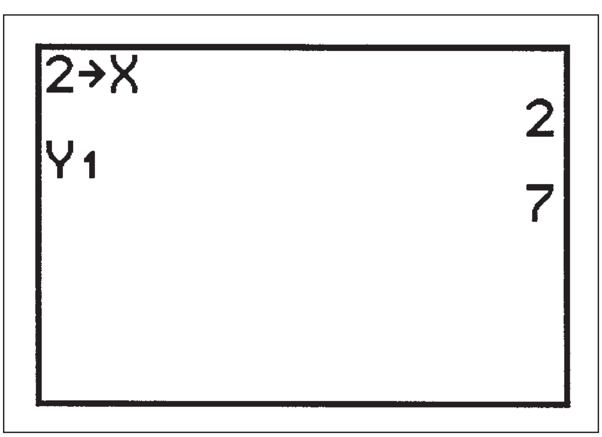



**Fig. 6 Calculator display after one round of Guess the Calculator's Rule**

The Function Machine gives students a visual image of an input-output device. The teacher may also use a symbolic representation of the function rule if desired. The Function Machine connects quite naturally with Guess My Rule. **Figure 7** shows a Function Machine representation for the function rule in the previous example. I have also found the Function Machine to be useful for teaching functions in college-level courses, particularly for introducing college students to the concept of composition of functions.

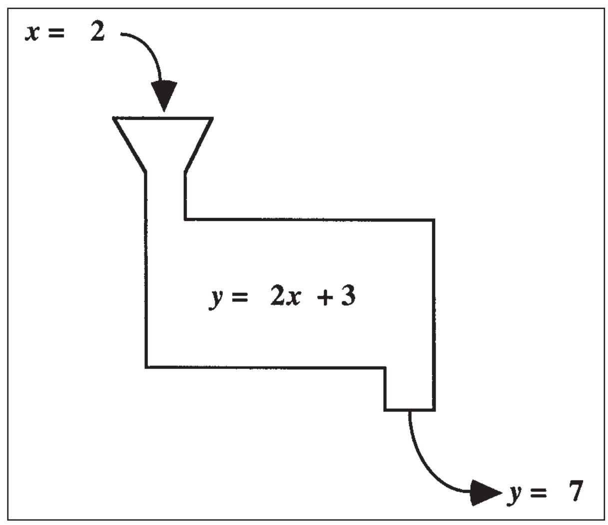



**Fig. 7 A Function Machine representation of $y = 2x + 3$**

**Properties of numbers**

Middle school students should have sufficient competence in arithmetic to thoughtfully explore the special relationships of 0 and 1 to the rest of the real-number system and the existence of opposites and reciprocals of numbers. In fact, such work is probably already part of most mathematics curricula in the middle grades, and it is an important precursor to algebra because these concepts are precisely the tools needed to solve first-degree equations.

Students in the middle grades should investigate and articulate the special nature of 0 with respect to both addition and multiplication and of 1 with respect to multiplication. Zero should be named as the additive identity, and 1, as the multiplicative identity. Students may also be able to recognize the appropriateness of the term *identity*. That is, when adding 0 to any number or when multiplying any number by 1, the identity of the other number is preserved.

Next students should be challenged to "make 0" by adding some number to a given number or to "make 1" by multiplying a given number by some number. Such activities will help them discover opposites and reciprocals, or multiplicative inverses. Students can also be challenged to find a reciprocal for 0. Once they realize the impossibility of this task, they might be able to articulate the fact that 0 has no reciprocal, or multiplicative inverse, and perhaps even state a mathematical argument in support of their assertion.

Finally, students should come away from their study of mathematics in the middle grades with an understanding and appreciation of the associative, commutative, and especially, the distributive properties of numbers. By the time students have reached the middle grades, they are using each of these properties in performing arithmetic computations. Their ability to apply these properties in the subsequent formal study of algebra will be enhanced if the uses of these properties in arithmetic are made explicit.

For example, the use of the commutative and associative properties should be emphasized when students are taught how to regroup sums to make computations easier. In **figure 8,** for example, when the order of 18 and 27 is reversed, students should be aware that they are using the commutative property; when 23 and 27 are grouped, as well as 18 and 32, students should know that they have applied the associative property.

Students who are well grounded in the commutative property, for example, should be able to fashion an argument that $x2 = 2x$. This assertion could also be supported using an area model for multiplication, as shown in **figure 9.** Students might draw such a model or build one using manipulatives.

$23 + 18 + 27 + 32$
$23 + 27 + 18 + 32$
$(23 + 27) + (18 + 32)$
$50 + 50$
$100$

**Fig. 8 Using the associative and commutative properties to simplify computation**

$$x2 = 2x$$

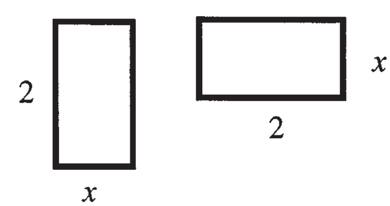



**Fig. 9 An area argument to verify the commutative property of multiplication**

In my experience, many beginning algebra students have difficulty using the distributive property to find such products as $3(x + 5)$ or $(x + 5)(x + 2)$. The difficulty arises despite the fact that these same students use the distributive property quite proficiently in calculating such products as $25 \times 7$ and $15 \times 12$. The problem is that the algorithm that they have learned so well was designed to be efficient, not necessarily to foster understanding. In the former products, reducing to a bare minimum the number of symbols that must be written impedes students' understanding of why and how the algorithm that they are applying works. Students in the early middle grades (and probably even earlier than that) should have numerous experiences in decomposing such products by applying the distributive property (see **fig. 10**). Sufficient experience in decomposing similar products should make the use of the distributive property more explicit for students and, in turn, should better prepare them to extend their use of the distributive property to products of polynomials (see **fig. 11**).

## Concluding Thoughts

SOME TIME AGO, I WAS DISCUSSING WAYS THAT the teacher education department in which I work has collaborated with other departments across the university. I happened to mention that I had cotaught a section of abstract algebra with a colleague from the mathematics department. Another of the teacher educators in the group, one whose area of specialization is not mathematics, commented that he found *any* algebra to be abstract. We all had a laugh about that one, but on reflection, I have no doubt that the school algebra he experienced *was* abstract—too abstract, at least, for his readiness to comprehend it at the time that he studied the subject.

I have suggested ways in which students' introductory encounters with algebra and algebraic thinking in middle school could be made more meaningful for them. Some of these activities involve using specialized materials, such as algebra manipulatives or graphing calculators. Others involve nothing more than fine-tuning our approaches to standard middle school topics. All these activities will help students form a strong conceptual base on which a more formal study of algebra may later be built.

| | |
|---|---|
| $25 \times 7$<br>$= (20 + 5) \times 7$<br>$= (20 \times 7) + (5 \times 7)$<br>$= 140 + 35$<br>$= 175$ | $15 \times 12$<br>$= (10 + 5) \times (10 + 2)$<br>$= (10 \times 10) + (10 \times 2) + (5 \times 10) + (5 \times 2)$<br>$= 100 + 20 + 50 + 10$<br>$= 180$ |

**Fig. 10 Decomposing arithmetic products using the distributive property**

| | |
|---|---|
| $3(x + 5)$<br>$3 \bullet x + 3 \bullet 5$<br>$3x + 15$ | $(x + 5)(x + 2)$<br>$(x \bullet x) + (x \bullet 2) + (5 \bullet x) + (5 \bullet 2)$<br>$x^2 + 2x + 5x + 10$<br>$x^2 + 7x + 10$ |

**Fig. 11 Decomposing algebraic products using the distributive property**

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